# Lecture Notes On Differential Equation

B.Sc. Semester -2
Mathematics

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# Contents

# Chapter 1

# Introduction to Differential Equations

The following topics are to be covered from differential equation of first order and first degree. Topics included here are from unit-3 of the syllabus according to choice base credit system effective from June-2010. The course code of the M-101 and title of the paper is *Geometry and calculus*.
**Differential Equations of First Order and First Degree:** Definition and method of solving of *homogeneous differential equations*, Definition and method of solving of *Linear differential equations of first order and first degree*, Definition and method of solving of *Bernoulli's differential equation* and Definition and methods of solving of *Exact differential equation.* **Differential Equations of First order and Higher Degree:** Differential equations of first order and first degree *solvable for x, solvable for y, solvable for p. Clairaut's form* of differential equation and *Lagrange's form* of differential equations.

**Definition 1.1.** Differential equation *is an equation which involves differentials or differential coefficients. For example,*

1. $\frac{dy}{dx} = x^2 + 2y.$

2. $r^2 \frac{d^2\theta}{dr^2} = a.$ *Where $a$ is constant.*

3. $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E\sin\omega t.$

**Definition 1.2.** *A differential equation is said to be* linear in dependent variable *if,*

1. *dependent variable and all its derivatives present are in first degree.*

2. *dependent variable and its derivatives are not multiplies together.*

3. *dependent variable and its derivatives are not multiplied with itself.*

4. *no transcedental functions of dependent variable and/or its derivative occur.*

**Remark 1.3.** *A differential equation which is not linear is said to be* Non-linear. *It is nice exercise to find out some examples of linear and non linear differential equation. You can check from examples given in the exercises. (do it!)*

**Definition 1.4.** *An* ordinary differential equation (O. D. E.) *is a differential equation which involves only ordinary derivatives.*

**Definition 1.5.** *A* partial differential equation (P. D. E) *is a differential equation which involves only partial derivatives. For example,*

1. $\frac{\partial U}{\partial t} = c\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$.

2. $\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}$.

**Definition 1.6.** *The* order of the differential equation *is defined to as the order of the highest derivative involved in the differential equation. Also, the* degree of the differential equation *is defined as the degree of the highest derivative involved in the differential equation, where all derivatives occurring therein are free from radicals and fraction.*

**Examples 1.7.** *(1) Decide the order and degree of the differential equation given by*

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \int 3dx = \sin x.$$

*Solution: The given differential equation is not free from integration sign. So, to decide order of a differential equation we have to differentiate with respect to x on both sides and make it free from integration.*

$$\Longrightarrow x^2 \frac{d^3 y}{dx^3} + 3x\frac{d^2 y}{dx^2} + \frac{dy}{dx} + 3 = \cos x.$$

*Here, order of the highest derivative involved is three. Therefore, order of differential equation is* $3$*, and degree of highest derivative is* $1$*. Thus, order is* $3$ *and degree is* $1$*.*
*(2)* $\sqrt[4]{(y'')^5} = \sqrt{7+3(y')^2}$
*Solution: To obtain degree of differential equation we have make differential equation free from radicals.*

$$\therefore (\sqrt[4]{(y'')^5})^4 = (\sqrt{7+3(y')^2})^4.$$

$$(y'')^5 = (7+3(y')^2)^2.$$

$$\left(\frac{d^2 y}{dx^2}\right)^5 = \left[7+3\left(\frac{dy}{dx}\right)\right]^2$$

*Which shows that order of the given differential equation is* $2$ *and degree is* $5$*.*

**Definition 1.8.** *1. A* solution *or* integral *or* primitive *of a differential equation is a relation between the variables which does not involve any derivatives and also satisfies given differential equation. For example,* $y = c_1 \cos x + c_2 \sin x$*, where* $c_1$ *and* $c_2$ *are arbitrary constants, is a solution of the differential equation given by* $\frac{d^2y}{dx^2} + y = 0$.

2. *A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the* general solution *or* complete integral *or* complete primitive.

3. *The solution obtained from the general solution by giving particular values to the arbitrary constants is called* particular solution. *For example,* $y = x^4 + 2$ *is a particular solution of the differential equation* $\frac{dy}{dx} = 4x^3$*, where* $c = 2$.

4. *A solution which can not be obtained from a general solution is called* singular solution . *For example,* $y = x\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2$. *The general solution is given by* $y = cx + 2c^2$*, where* $c$ *is an arbitrary constant. Also,* $8y = x^2$ *is a singular solution which can not be obtained by putting any value of* $c$.

**Examples 1.9.** *(1) Find the differential equation from* $y = ax - a^2$*, where* $a$ *is an arbitrary constant.*
*Solution: Differentiating* $y = ax - a^2$ *with respect to* $x$ *we get* $\frac{dy}{dx} = a$. *Substituting we get desired differential equation* $y = \left(\frac{dy}{dx}\right)x - \left(\frac{dy}{dx}\right)^2$.
*(2)Form the differential equation from* $y = Ae^{2x} + Be^{5x}$*; where* $A$ *and* $B$ *are arbitrary constants.*
*Solution: Here, two arbitrary constants* $A$ *and* $B$ *are present, therefore to eliminate them we have to differentiate two times.*

$$\therefore \frac{dy}{dx} = 2Ae^{2x} + 5Be^{5x}. \tag{1.1}$$

*again by differentiating with respect to* $x$ *we get,*

$$\therefore \frac{d^2y}{dx^2} = 4Ae^{2x} + 25Be^{5x}. \tag{1.2}$$

*Multiply equation* $y = Ae^{2x} + Be^{5x}$ *by* $-2$ *and adding in (4.2) we get*

$$\frac{dy}{dx} - 2y = 3Be^{5x} \implies Be^{5x} = \frac{1}{3}\left[\frac{dy}{dx} - \frac{2}{3}y\right]. \tag{1.3}$$

*Now multiply (4.1) by* $-5$ *and adding in* (4.2) *we get,* $Ae^{2x} = \frac{5}{6}\frac{dy}{dx} - \frac{1}{6}\frac{d^2y}{dx^2}$. *Thus by substituting values of constants we get*

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0.$$

*Which is required differential equation.*

**Exercise-I**

Que-1. Find the differential equation from the following equations.

1. $xy = ce^x + be^{-x} + x^2$, where $b$ and $c$ are arbitrary constants.

2. $ax^2 + by^2 = 1$, where $a$ and $b$ are arbitrary constants.

3. $y = ax + bx^2$, where $a$ and $b$ are arbitrary constants.

4. $r^2 = a^2 cos2\theta$, where $a$ is an arbitrary constant.

Que-2. Find out order and degree of the following differential equations.

1. $x^2 \frac{d^2y}{dx^2} - x(\frac{dy}{dx})^3 + y = cosx.$

2. $\frac{y'}{y} = \frac{d}{dx}[\frac{y''}{y'}].$

3. $(\frac{dy}{dx})^2 = \sqrt{1 + (\frac{dy}{dx})^2}.$

4. $\frac{d^2y}{dx^2} = 3\frac{dy}{dx} + \int xdx.$

Que-3. Show that $y = e^{2x}$ is a solution of a differential equation

$$3x^2 \frac{d^2y}{dx^2} + 2(1 - 3x^2)\frac{dy}{dx} - 4y = 0.$$

Que-4. Prove that $y = 2x + 5e^{-x}$ is a particular solution of a differential equation

$$(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Que-5. Which curve is represented by a differential equation

$$2a\frac{d^2y}{dx^2} = 1?$$

# Chapter 2

# Differential Equations of First Order and First Degree.

In order to solve the differential equation, we need to investigate, whether the solution exists. It is not always possible to find a real analytic solution of a given differential equation. For example, $\left(\frac{dy}{dx}\right)^2 = -5$ has no solution for any real value of $y$. In our case we shall discuss some of the special types of differential equations for which analytic solution exists. Only those differential equations which belong to or can be reduced to any one of the following type can be solved by standard procedure. These types are,

1. Differential equation in which variables are separable.

2. Homogeneous differential equations.

3. Nonhomogeneous differential equations which can be reduced to homogeneous differential equations.

4. Linear differential equations.

5. Bernoulli's differential equations. These are nonlinear types of differential equations which can be reduced to linear form.

6. Exact differential equations.

## 2.1 Differential equations in which variables are separable.

The general form of this type of equation is

$$M(x)dx + N(y)dy = 0, \tag{2.1}$$

which can be solved by direct integration as $\int M(x)dx + \int N(y)dy = c$, where $c$ is an arbitrary constant. If the differential equation is given in the form

$$f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0, \tag{2.2}$$

then we can reduce it in the form of equation (2.1) by rewriting as

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0,$$

provided $f_2(x) \neq 0$, $g_1(y) \neq 0$. Also, if the given differential equation is in the form

$$\frac{dy}{dx} = f(ax + by + c), \tag{2.3}$$

then put $ax + by + c = u$, to convert it in general form. Let us see following examples to understand this method well.

**Examples 2.1.** *1.* $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

*Solution: The given differential equation is not in its general form. In order to solve the given differential equation first we will convert it into general form.*

$$\frac{dy}{dx} = e^{-2y}(e^3x + x^2)$$
$$\implies e^{2y}dy = (e^{3x} + x^2)dx$$
$$\implies (e^{3x} + x^2)dx - e^{2y}dy = 0,$$

*which is in the general form and hence the solution can be obtained by direct integration.*

$$\implies \int (e^{3x} + x^2)dx - \int e^{2y}dy = c$$
$$\implies \frac{e^{3x}}{3} + \frac{x^3}{3} - \frac{e^{2y}}{2} = c$$
$$or\, 3e^{2y} = 2(e^{3x} + x^3) + c'.$$

*Which is a general solution of the given differential equation and $c'$ is an arbitrary constant.*

2. *Obtain particular solution of* $\frac{dy}{dx} = (4x+y+1)^2$, *where* $y(0) = 1$
   *Solution: The given differential equation is not of the form of separable variable. Hence, to convert it into separable variable form we put* $4x+y+1 = t$ *and* $\frac{dt}{dx} = 4 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{dt}{dx} - 4$. *Put these values in equation we get*

$$\frac{dt}{dx} - 4 = t^2$$

$$\therefore \frac{dt}{t^2+4} = dx.$$

$$\int \frac{dt}{t^2+4} = \int dx + c, \text{ where } c \text{ is an arbitrary constant.}$$

$$\therefore \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\therefore \frac{1}{2}\tan^{-1}\frac{4x+y+1}{2} = x + c$$

*Put* $x = 0$ *and* $y = 1$ *we get* $\tan(2c) = 1 \implies 2c = \frac{\pi}{4}$. *Thus, particular solution is given by*

$$4x+y+1 = 2\tan\left(2x + \frac{\pi}{4}\right).$$

## 2.2 Homogeneous differential equations

**Definition 2.2.** *Let* $E \subset \mathbb{R}^2$. *A function* $f : E \to \mathbb{R}$ *is said to be* homogeneous of degree $n$ *if it can be written in the form* $f(x,y) = x^n \phi(\frac{y}{x})$.

**Definition 2.3.** *A differential equation is said to be* homogeneous differential equation *if it is of the form*

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \text{ or } \frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}. \tag{2.4}$$

*Where* $P(x,y)$ *and* $Q(x,y)$ *are homogeneous functions of equal degree in variables* $x$ *and* $y$.

In order to solve homogeneous differential equations we need to follow mainly three following steps.

1. Put $y = vx$ in the given differential equation and evaluate $\frac{dy}{dx}$.

2. Substitute the values of $y$ and $\frac{dy}{dx}$ in main equation and bring the equation in the form of separable variable.

3. Solve by the method of separable variable.

**Examples 2.4.** 1. *Solve:* $(x^2+y^2)dx - 2xydy = 0$
   *Solution:*

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{(1+\frac{y}{x})}{\frac{2y}{x}} \tag{2.5}$$

*Put $y = vx$ we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substitute these values in equation (2.5) we get,*

$$v + x\frac{dv}{dx} = \frac{1+v}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\therefore \frac{2v}{1-v^2}dv = \frac{1}{x}dx$$

*Which is now in the separable variable form. So, solution can be obtain by direct integration. Integrating both side we get,*

$$\therefore \int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$$

$$\therefore -\log(1-v^2) = \log x + \log c \text{ where } c \text{ is an arbitrary constant.}$$

$$\therefore \log x + \log(1-v^2) = \log c', \text{ where } c' = c^{-1}$$

$$\therefore \log(x(1-v^2)) = \log c'$$

*by taking exponential on both sides we get,*

$$x(1-v^2) = c',$$

*now substitute the value of $v$ in above equation, we get*

$$x^2 - y^2 = c'x$$

*which is the general solution of the given differential equation.*

## 2.3 Nonhomogeneous differential equations which can be reduced to homogeneous differential equations.

A differential equation of the form,

$$\frac{dy}{dx} = \frac{ax+by+c}{lx+my+n} \tag{2.6}$$

is not homogeneous differential equation, but by making some change we can reduce it to the case of homogeneous differential equation.

**Case-I** $\frac{a}{l} \neq \frac{b}{m}$. In order to solve differential equation having this case, let $x = x' + h$ and $y = y' + k$, where $h$ and $k$ are constants.Also, $dx = dx'$ and $dy = dy'$. Then equation (2.6) reduces to

$$\frac{dy'}{dx'} = \frac{ax'+by'+ah+bk+c}{lx'+my'+lh+mk+n} \tag{2.7}$$

In this equation we select $h$ and $k$ by solving $ah + bk + c = 0$ and $lh + mk + n = 0$ such that equation (2.7) will turn out to homogeneous differential equation $\frac{dy'}{dx'} = \frac{ax'+by'}{lx'+my'}$, where $al - bm \neq 0$. Which is homogeneous in the variables $x'$ and $y'$. So solve it by putting $y' = vx'$.

**Case-II** $\frac{a}{l} = \frac{b}{m}$. In this case $al - bm = 0$,and hence $h$ and $k$ will be indetermined or infinity. Hence put $\frac{a}{l} = \frac{b}{m} = t$, where $t$ is constant in equation (2.6) we get

$$\frac{dy}{dx} = \frac{(lx+my)t+c}{(lx+my)+n}. \tag{2.8}$$

Now by substitute $lx + my = t$ in equation (2.8) we can solve the given differential equation. Let us see the following examples to understand this method well.

**Examples 2.5.** *(1)*$\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$. *Solution: The differential equation is given by*

$$\frac{dy}{dx} = \frac{y+x-2}{y-x-4} \tag{2.9}$$

*is not homogeneous differential equation. By comparing with (2.6) we get* $a = 1, b = 1, l = -1, m = 1$. *Here,* $\frac{a}{l} = -1 \neq \frac{b}{m} = 1$. *Hence substitute* $x = x' + h$ *and* $y = y' + k$ *in equation (2.9) we get,*

$$\frac{dy'}{dx'} = \frac{y'+x'+(k+h-2)}{y'-x'+(k-h-4)} \tag{2.10}$$

*To convert equation (2.10) in homogeneous differential equation we take* $k+h-2 = 0$ *and* $k-h-4 = 0$, *by solving we get* $h = -1, k = 3$. *Hence with these values of h and k equation (2.10) reduces to,*

$$\frac{dy'}{dx'} = \frac{y'+x'}{y'-x'}, \text{ which is homogeneous differential equation.} \tag{2.11}$$

*In order to solve put* $y' = vx'$ *and* $\frac{dy}{dx} = v + x'\frac{dv}{dx'}$ *in equation (2.11) we obtain,*

$$v + x'\frac{dv}{dx'} = \frac{vx'+x'}{vx'-x'} = \frac{v+1}{v-1}$$

$$\therefore x'\frac{dv}{dx'} = \frac{v+1}{v-1} - v - \frac{1+2v-v^2}{v-1}$$

$$\therefore \frac{v-1}{1+2v-v^2}dv = \frac{dx'}{x'}, \text{ which is separable variable form}$$

*By integrating term by term we get,*

$$\int \frac{v-1}{1+2v-v^2}dv = \int \frac{dx'}{x'} + c, \text{ where } c \text{ is an arbitrary constant.}$$

$$\therefore -\frac{1}{2}\int \frac{2-2v}{1+2v-v^2dv} = \log x' + c$$

$$\therefore \log\left(1+2\frac{y'}{x'} - \frac{y'^2}{x'^2}\right) + \log x'^2 = -2c$$

$$\therefore \log(x'^2 + 2x'y' - y'^2) - \log x'^2 + \log x'^2 = -2c$$
$$\therefore x'^2 + 2x'y' - y'^2 = e^{-2c} = c'$$

*by substituting* $x' = x+1$ *and* $y' = y-3$, *we get* $x^2+2xy-y^2-4x+8y-14 = c'$, *which is general equation of given differential equation.* *(2)*$(x-y+2)dx + (2x-2y-4)dy = 0$
*Solution: The differential equation is given by,*

$$\frac{dy}{dx} = -\frac{x-y+2}{2(x-y)-4} \tag{2.12}$$

*is not homogeneous differential equation. By comparing with (2.6) we get* $a = -1, b = l, l = 2, m = -2$. *Here,* $\frac{a}{l} = -\frac{1}{2} = \frac{b}{m}$. *Therefore h and k can not be determined. Put* $x - y = z$ *and* $1 - \frac{dy}{dx} = \frac{dz}{dx}$ *in equation (2.12) we get,*

$$1 - \frac{dz}{dx} + \frac{z+2}{z-4} = 0$$
$$\therefore \frac{dz}{dx} + \frac{3z-2}{2z-4} = 0$$
$$\therefore \frac{2z-4}{3z-2}dz = dx, \text{ which is separable variable form.}$$

*In order to get solution integrate the terms separately we get*

$$\int \frac{2z-4}{3z-2}dz = \int dx + c, \text{ where } c \text{ is an arbitrary constant}$$
$$\therefore \int \frac{2}{3}\frac{3z-2-4}{3z-2}dz = \int dx + c$$
$$\therefore \frac{2}{3}\int \left(1 - \frac{4}{3z-2}\right)dz = x + c$$
$$\therefore \frac{2}{3}\left[x - y - \frac{4}{3}\log[3(x-y)-2]\right] = 3x + c', \text{ where } c' = 3c$$
$$\therefore x + 2y + \frac{8}{3}\log[3(x-y)-2] + c', \text{ which is a general solution.}$$

**Exercise-II**

Identify type of the following differential equations and solve them.

1. $2y\frac{dy}{dx} = x^2 + \sin 3x$. (Ans: $3y^2 = x^3 - \cos 3x + c$.)
2. $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$. (Ans: $\tan y = c(1-e^x)^3$.)
3. $\frac{y}{x}\frac{dy}{dx} + \frac{2(x^2+y^2)-1}{x^2+y^2+1} = 0$. (Ans: $2x^2 + y^2 + 3\log(x^2+y^2-2) = c$.)
4. $x^4\frac{dy}{dx} + x^3y + cosec(xy) = 0$. (Ans: $\cos xy + \frac{1}{2x^2} = c$.)

5. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$. (Ans: $(x+a)(1-ay) = cy$.)
6. $x\frac{dy}{dx} = y + \cos^2\left(\frac{y}{x}\right)$. (Ans: $\tan\left(\frac{y}{x}\right) = \log|cx|$
7. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$. ( Ans: $y = x\log y + cx$).
8. $y - x\frac{dy}{dx} = \sqrt{y^2 - x^2}$. ( Ans: $y + \sqrt{y^2 - x^2} = c$.)
9. $\frac{x+y+1}{x-y+1}$. ( Ans: $\tan^{-1}\frac{y}{x+1} = \log\left(c\sqrt{(x+1)^2 + y^2}\right)$).
10. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$. ( Ans: $(x+y-2)(x-y)^{-3} = c$).
11. $(3y+2x+4)dx - (4x+6y+5)dy = 0$. ( Ans: $21x - 42y + 9\log(14x + 21y + 22) = c'$).
12. $(2x+9y-20)dx = (6x+2y-10)dy$. ( Ans: $(y-2x)^2 = c(x+2y-5)$).

## 2.4 Linear differential equations.

**Definition 2.6.** *A differential equation of the form* $\frac{dy}{dx} + Py = Q$*, where* $P$ *and* $Q$ *are either constants or functions of* $x$ *is said to be* linear differential equation of first order. *For example,* $\frac{dy}{dx} + (\sec^2 x)y = \sec^2 x \tan x$ *is linear differential equation of first order.*

In order to solve the linear differential equation we use the method of separable variable. Linear differential equation of first order is given by

$$\frac{dy}{dx} + Py = Q, \text{ where P and Q are either constants or functions of } x. \tag{2.13}$$

First we solve $\frac{dy}{dx} + Py = 0$ by using separable variable method. For

$$\int \frac{dy}{y} = -\int P dx + c. \text{ where } c \text{ is an arbitrary constant.}$$

$$\log y = -\int P dx + c'.$$

$$\therefore y = e^{-\int P dx} e^{-c'}.$$

$$\therefore y = e^{-\int P dx} c.$$

Now differentiate on both sides with respect to $x$ we get,

$$e^{\int P dx}\frac{dy}{dx} + ye^{\int P dx}P = 0.$$

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = 0.$$

$$\therefore \frac{d}{dx}\left(ye^{\int P dx}\right) = e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = 0. \tag{2.14}$$

Since $e^{\int P dx} \neq 0$ we multiply equation (2.13) by $e^{\int P dx}$ on both sides we get

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P dx}.$$

$$\therefore \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}.$$

By integrating on both sides we have

$$\int \frac{d}{dx}(ye^{\int P dx})dx = \int Qe^{\int P dx}dx + c.$$

$\therefore ye^{\int P dx}dx = \int Qe^{\int P dx}dx+c,$ where $c$ is an arbitrary constant. Which is the general solution of the given differentia

**Remark 2.7.** *Here we can solve the equation by multiplying the given differential equation by $e^{\int P dx}$ and hence we call $e^{\int P dx}$ an* integrating factor *denoted by* I. F *then here $I.F = \int e^{Pdx}$. Therefore the general formula for finding the solution of linear differential equation is given by*

$$y(I.F.) = \int Q(I.F.)dx + c.$$

**Examples 2.8.** *(1) Solve:* $(x+1)\frac{dy}{dx} + 2y = 1.$
*Solution: To convert the given differential equation in general form of the linear differential equation we divide both side by* $(x+1)$.

$$\therefore \frac{dy}{dx} + \frac{2}{x+1}y = \frac{1}{x+1}.$$

*Compare this with equation (2.13) we get $P = \frac{2}{x+1}$ and $Q = \frac{1}{x+1}$.*

$$\therefore e^{\int P dx} = e^{\int \frac{2}{x+1}dx} = e^{2\log(x+1)} = (x+1)^2.$$

*Now we know the general formula for finding the solution of differential equation is*

$$ye^{\int P dx} = \int Qe^{\int P dx}dx.$$

*By substitutes values we get*

$$y(x+1)^2 = \int \frac{1}{1+x}(1+x)^2 dx + c.$$

$$y(x+1)^2 = \int (x+1)dx + c = \frac{x^2}{2} + x + c.$$

$$y(x+1)^2 = \frac{x^2}{2} + x + c. \text{ Which is a general solution.}$$

*(2) Solve:* $(1+y^2)dx = (\tan^{-1} y - x)dy.$

*In the given differential equation the term containing $x$ is $1$ with degree $1$. Therefore the equation can be converted to a differential equation which is linear in $x$ given by $\frac{dx}{dy} + Px = Q$.*

$$\therefore \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1} y}{1+y^2}$$

*Comparing this equation with general form we get, $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1} y}{1+y^2}$.*

$$\therefore I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

*Now put this value in general formula given by $xe^{\int P dy} = \int Qe^{\int P dy} dy$ we get*

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy + c$$

*where $c$ is an arbitrary constant. Now for right hand side integration we take $\tan^{-1} y = t$, $\frac{dy}{1+y^2} = dt$ we get*

$$\therefore xe^{\tan^{-1} y} = \int te^t dt + c.$$

*By integrating by parts we get*

$$xe^{\tan^{-1} y} = te^t - \int 1e^t dt + c.$$

$$\therefore xe^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + c$$

*which is a general solution.*

## 2.5 Bernoulli's differential equations.

**Definition 2.9.** *A differential equation of the form $\frac{dy}{dx} + Py = Qy^n, n \in \mathbb{R} \setminus \{0\}$ is said to be* Bernoulli's differential equation

In order to solve Bernoulli's differential equation we will use the method of solving linear differential equation. Bernoulli's differential equation is given by

$$\frac{dy}{dx} + Py = Qy^n, n \in \mathbb{R} \setminus \{0\}. \tag{2.15}$$

Divide both sides by $y^n$ we get $y^{-n}\frac{dy}{dx} + y^{1-n}P = Q$. Now multiply by $(1-n)$ both sides we get

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)y^{1-n}P = (1-n)Q. \tag{2.16}$$

Now put $v = y^{(1-n)}$ and $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ in equation (2.16) we get

$$\frac{dv}{dx} + (1-n)Pv = (1-n)Q \tag{2.17}$$

Which is linear in variable $v$ and can be solved by method of linear differential equation. Hence substitute

$$\therefore I.F. = e^{\int Pdx} = e^{\int (1-n)Pdx}$$

in equation $ve^{\int Pdx} = \int Qe^{\int Pdx} + c$

$$\therefore ve^{\int (1-n)Pdx} = \int (1-n)Qe^{\int (1-n)Pdx} dx + c$$

$$\therefore y^{1-n}e^{\int (1-n)Pdx} = \int (1-n)Qe^{\int (1-n)Pdx} dx + c.$$

where $c$ is an arbitrary constant . Which is a general solution.

**Examples 2.10.** *(1)Solve:* $x\frac{dy}{dx} + y = x^3y^6$
*Solution: The given differential equation is not linear in $x$ also not linear $y$. To convert it into Bernoulli's form we divide the equation by $xy^6$ we get*

$$y^{-6}\frac{dy}{dx} + y^{-5}\frac{1}{x} = x^2. \tag{2.18}$$

$\therefore$ *put $y^{-5} = v$ and $-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$ in equation (2.18) we get $\frac{dv}{dx} - \frac{5}{x}v = -5x^2$ which is linear in $v$. Hence comparing with general form of linear differential equation we get $P = -\frac{5}{x}$ and $Q = -5x^2$. Now*

$$I.F. = e^{\int Pdx} = e^{\int \frac{-5}{x}dx} = x^{-5}.$$

*Now formula for solution is given by*

$$ve^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$

*where $c$ is an arbitrary constant.*

$$\therefore y^{-5}x^{-5} = \int -5x^2x^{-5} dx + c$$

$$\therefore y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c, \textit{where } c \textit{ is an arbitrary constant. Which is a general solution.}$$

*(2) Solve:* $x\frac{dy}{dx} - y = y^2 \log x$.
*Solution: To convert this equation in form of Bernoulli's differential equation we divide both sides by $x$ we get*

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{\log x}{x}y^2.$$

*Now comparing with the general form of Bernoulli's differential equation $\frac{dy}{dx} + Py = Qy^n$, we get $P = -\frac{1}{x}; Q = \frac{\log x}{x}$ with $n = 2$. Therefore the solution is given by*

$$y^{1-n}e^{\int (1-n)Pdx} = \int (1-n)Qe^{\int (1-n)Pdx} + c.$$

$$\therefore y^{-1}x = \int -1\frac{\log x}{x}x dx + c.$$

$$\therefore -\int \log x dx + c \implies -[\log x x - \int \frac{1}{x} x dx] + c.$$

$\therefore x = y(c + x - x\log x)$. *Which is a general solution of the given differential equation.*

**Remark 2.11.** *The general form of Bernoulli's differential equation* $\frac{dy}{dx} + Py = Qy^n$; $n \in \mathbb{R} \setminus \{0\}$ *is given by*

$$f'(y)\frac{dy}{dx} + f(y)P = Q.$$

*In order to solve this we put* $u = f(y)$ *we get* $\frac{du}{dx} = f'(y)\frac{dy}{dx}$ *in general form we get* $\frac{du}{dx} + Pu = Q$, *which is linear differential equation. Let us see the following examples to understand.*

**Examples 2.12.** *(1) Solve:* $\sin y\frac{dy}{dx} + x\cos y = x$.
*Solution: Here* $u = \cos y$ *and* $\frac{du}{dx} = -\sin y\frac{dy}{dx}$. *Substitute these values in given differential equation we get*

$$\frac{du}{dx} - xu = -x.$$

*Which is linear differential equation in variable* $v$. *Therefore solution is given by*

$$u(I.F.) = \int Q(I.F.)dx + c.$$

$$ue^{\frac{-x^2}{2}} = \int (-x)e^{\frac{-x^2}{2}}dx + c.$$

$$\cos y = \frac{1}{2} + ce^{\frac{x^2}{2}}.$$ *Which is a general solution.*

*(2) Solve:* $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x}(\log y)^2$.
*Solution: Divide both sides by* $y$ *we get*

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = \frac{1}{x}(\log y)^2.$$

*Now put* $u = \log y$, *we get* $\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}$. *Substitute these values in above equation we get*

$$\frac{du}{dx} + \frac{u}{x} = \frac{u^2}{x} \implies \frac{1}{u^2}\frac{du}{dx} + \frac{1}{x}\frac{1}{u} = \frac{1}{x}$$

. *Which is in the form of Bernoulli's differential equation. By putting* $\frac{1}{u} = t$ *and solving it we get* $(\log y)^{-1} = 1 + cx$ *which is general solution of given differential equation.*

**Exercise-III**

Identify type of the following differential equations and solve them.

1. $\frac{dy}{dx} + y\cos x = \sin x \cos x$ (Ans: $y = \sin x + ce^{-\sin x} - 1$.)

2. $\frac{dy}{dx} + 2xy = 2x$, also $y = 3$ when $x = 0$ obtain a particular solution. (Ans: $y = 1 + ce^{-x^2}$ and P.S. is$y = 1 + 2e^{-x^2}$.)

3. $\frac{dy}{dx} + y\tan x = \sec x$. (Ans: $y = \sin x + c\cos x$.)

4. $\cos^2 x \frac{dy}{dx} + y = \tan x$. (Ans: $y = \tan x - 1 + ce^{-\tan x}$.)

5. $(1 + x^2)dy = (\tan^{-1} x - y)dx$. (Ans: $y = \tan^{-1} x - 1 + ce^{-\tan^{-1} x}$.)

6. $x\frac{dy}{dx} + 2y = x^2 \log x$. (Ans: $y = \frac{x^2}{4}\log x - \frac{x^2}{16} + cx^{-2}$.)

7. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$. (Ans: $y\sin x = -5e^{\cos x + c}$.)

8. $\frac{dy}{dx} + 2y\tan x = \sin x$,also obtain particular solution with $y = 0$ when $x = \frac{\pi}{3}$. (Ans: $y\sec^2 x = \sec x + c$; P.S $= y\sec^2 x = \sec x - 2$)

9. $(x + 2y^3)\frac{dy}{dx} = y$. (Ans: $x = y^3 + cy$.)

10. $x\log x\frac{dy}{dx} + y = 2\log x$. (Ans: $y\log x = (\log x)^2 + c$.)

11. $\frac{dy}{dx} + y\tan x = y^3 \sec x$. (Ans: $\cos^2 x = y^2(c + 2\sin x)$)

12. $xy(1 + xy^2)\frac{dy}{dx} = 1$. (Ans:$\frac{1}{x} = (2 - y^2) + ce^{\frac{-y^2}{2}}$.)

13. $\frac{dy}{dx} + y\tan x = \frac{\cos x}{y}$. (Ans: $y^2 = \cos^2 x\left[c + \log\tan\left(\frac{x}{4} + \frac{x}{2}\right)\right]$.)

14. $\sec^2 y\frac{dy}{dx} + x\tan y = x^3$. (Ans: $\tan y = x^3 - 3x^2 + 6x - 6 + ce^{-x}$.)

15. $(x^3y^3 + xy)dx = dy$. (Ans: $y^{-1} = 2 - x^2 + ce^{\frac{-x^2}{2}}$.)

16. $\frac{dy}{dx} + y\cos x = y^3 \sin 2x$. (Ans: $y^{-2} = 2\sin x + 1 + ce^{2\sin x}$.)

17. $x\frac{dy}{dx} = y - \sqrt{y}$. (Ans: $4c^2x = (y - 1 - c^2x)^2$.)

18. $x^3\frac{dy}{dx} - x^2y + y^4 = 0$. (Ans: $y^3(3x + c) = x^3$.)

19. $\frac{dy}{dx} + y\log y = xye^x$. (Ans: $x\log y = (x-1)e^x + c$.)

## 2.6 Exact differential equations.

**Definition 2.13.** *A differential equation* $M(x,y)dx + N(x,y)dy = 0$ *is said to be* exact *if there exists a function* $f(x,y)$ *such that* $d[f(x,y)] = Mdx + Ndy$. *That is,*

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy.$$

*In other words if a differential equation can be obtain by direct differentiation of its solution, then we call it an* exact differential equation.

**Necessary and Sufficient Condition for differential equation** $M(x,y)dx + N(x,y)dy = 0$ **to be exact:**

**Theorem 2.14.** *The necessary and sufficient condition for the differential equation* $M(x,y)dx+N(x,y)dy = 0$ *to be exact is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

*Where* $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ *denotes the partial derivatives of* $M$ *and* $N$ *with respect to* $y$ *and* $x$ *respectively.*

In order to solve an differential equation of the type $M(x,y)dx + N(x,y)dy = 0$, first check the condition of exactness, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If the condition satisfied, then the given differential equation is exact and solution is given by

$$\int_{y \text{ constant}} Mdx + \int \Big(\text{Terms in } N \text{ which are independent of x}\Big)\, dy = c.$$

Where $c$ is an arbitrary constant.

**Examples 2.15.** *(1)Solve:* $(x^2 - ay)dx + (y^2 - ax)dy = 0$.
*Solution: Here* $M(x,y) = x^2 - ay$ *and* $N(x,y) = y^2 - ax$

$$\therefore \frac{\partial M}{\partial y} = -a \text{ and } \frac{\partial N}{\partial x} = -a.$$

*Therefore the given differential equation is an exact differential equation. The solution is given by*

$$\int_{y \text{ constant}} Mdx + \int \Big(\text{Terms in } N \text{ which are independent of } x\Big)\, dy = c.$$

$$\therefore \int_{y \text{ constant}} (x^2 - ay)dx + \int y^2 dy = c$$

$$\therefore \frac{x^3}{3} - ayx + \frac{y^3}{3} = c.$$

$$x^3 + y^3 - 3axy = 3c. \text{ Which is a general solution.}$$

*(2) Solve:* $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0.$
*Solution: We write this equation in the form* $M(x,y)dx+N(x,y)dy=0$, *we get* $(y\cos x+\sin y+y)dx+(\sin x+x\cos y+x)dy=0$. *and also* $M(x,y)=y\cos x+\sin y+y$, $N(x,y)=\sin x+x\cos y+x$.

$$\therefore \frac{\partial M}{\partial y}=\cos x+\cos y+1=\frac{\partial N}{\partial x}.$$

*Therefore the given differential equation is an exact differential equation. The solution is given by*

$$\int_{y\ constant} M dx+\int \big(\text{ Terms in } N \text{ which are independent of } x\big)\, dy=c.$$

$$\therefore \int_{y\ constant} (y\cos x+\sin y+y)dx+\int 0dx=c.$$

$$\therefore y\sin x+x\sin y+yx=c. \text{ Which is a general solution .}$$

**Remark 2.16.** *If condition* $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, *then the given differential equation is not exact. In this case, if there exist some function* $f(x,y)$ *of two variables such that*

$$f(x,y)[M(x,y)dx+N(x,y)dy=0]$$

*become exact, then* $f(x,y)$ *is called an* integrating factor *denoted by I.F. For example, the differential equation* $x\frac{dy}{dx}+2y+3x=0$ *is not exact, but by multiplying with x we get*$x^2\frac{dy}{dx}+2yx+3x^2=0$ *which is an exact differential equation. Thus, here integrating factor is* $x$.

**Rules for Integrating factor for** $M(x,y)dx+N(x,y)dy=0$**:**

1. If $M(x,y)dx+N(x,y)dy=0$ is homogeneous differential equation with $Mx+Ny\neq 0$, then integrating factor will be $\frac{1}{Mx+Ny}$.
2. If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is only function of $x$ say $f(x)$, then $e^{\int f(x)dx}$ will be an integrating factor.
3. If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}$ is only function of $y$ say $g(y)$, then $e^{\int g(y)dy}$ will be an integrating factor.
4. If given differential equation is of the form $f_1(x,y)ydx+f_2(x,y)xdy=0$, then integrating factor will be $\frac{1}{Mx-Ny}$, where $Mx-Ny\neq 0$.

**Examples 2.17.** *(1) Solve:*$(x^2+y^2+2x)dx+2ydy=0.$
*Solution: Comparing the given differential equation with* $M(x,y)dx+N(x,y)dy=0$, *we get* $M(x,y)=x^2+y^2+2x$ *and* $N(x,y)=2y$. *Here* $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, *therefore the given differential equation is not exact. Notice that,* $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=1$ *which is only function of x say* $f(x)$. *Hence I.F.* $=e^{\int f(x)dx}=e^x$.

$$\therefore I.F[(x^2+y^2+2x)dx+2ydy=0] \text{ which is now reduced to an exact differential equation.}$$

*Now,* $e^x[(x^2+y^2+2x)dx+2ydy] = d((x^2+y^2)e^x) = 0$
*Thus, the solution is* $\int e^x[(x^2+y^2+2x)dx+2ydy] = \int d((x^2+y^2)e^x) = c$, *where* $c$ *is an arbitrary constant.* $\therefore (x^2+y^2)e^x = c$ *is a general solution.*
*(2) Solve:* $(xy\sin(xy)+\cos(xy))ydx+(xy\sin(xy)-\cos(xy))xdy = 0$.
*Solution: Comparing the given differential equation with* $M(x,y)dx+N(x,y)dy = 0$, *we get* $M(x,y) = (xy\sin(xy)+\cos(xy))y$ *and* $N(x,y) = (xy\sin(xy)-\cos(xy))x$. *Here* $\frac{\partial M}{\partial y} = x^2y^2\cos(xy)+yx\sin(xy)-yx\sin(xy)+\cos(xy) \neq x^2y^2\cos(xy)+3yx\sin(xy)-\cos(xy) = \frac{\partial N}{\partial x}$, *therefore the given differential equation is not exact. Notice that, it is of the form* $f_1(x,y)ydx+f_2(x,y)xdy = 0$, *therefore integrating factor will be* $\frac{1}{Mx-Ny} = \frac{1}{2xy\cos(xy)}$, *where* $Mx-Ny \neq 0$.

$$\therefore I.F.[M(x,y)dx+N(x,y)dy] = \frac{1}{2xy\cos(xy)}[(xy\sin(xy)+\cos(xy))ydx+(xy\sin(xy)-\cos(xy))xdy]$$

*is now reduced to exact differential equation. Thus, solution is given by,*

$$\int_{y\ constant} \frac{y}{2}\tan(xy)+\frac{1}{2x}dx-\int \frac{1}{2y}dy = \log c,$$

*where* $c$ *is an arbitrary constant.*

$$\therefore \frac{y}{2}\frac{\log\sec(xy)}{y}+\frac{1}{2}\log x-\frac{1}{2}\log y = \log c.$$

$$\therefore \log\sec(xy)+\log\frac{x}{y} = 2\log c$$

$$x = c'y\cos(xy),\ \textit{which is a general solution.}$$

*(3)Solve:* $x^2ydx-(x^3+y^3)dy = 0$.
*Solution: Comparing the given differential equation with* $M(x,y)dx+N(x,y)dy = 0$, *we get* $M(x,y) = x^2y$ *and* $N(x,y) = -(x^3+y^3)$. *Here* $\frac{\partial M}{\partial y} = x^2 \neq -3x^2 = \frac{\partial N}{\partial x}$, *therefore the given differential equation is not exact. Notice that given differential equation is homogeneous differential equation. Hence,* $I.F = \frac{1}{Mx+Ny} = \frac{-1}{y^4}$.

$$\therefore I.F.[M(x,y)dx+N(x,y)dy] = \frac{-1}{y^4}[x^2ydx-(x^3+y^3)dy]$$

*is now reduced to exact differential equation. The solution is given by*

$$\int_{y\ constant} \frac{-x^2}{y}dx+\int\frac{1}{y}dy = \log c,$$

*where* $c$ *is an arbitrary constant.*

$$\therefore \frac{-x^3}{3y^3}+\log y = \log c.$$

$$\therefore \log y = \log c+\frac{-x^3}{3y^3}$$

$$\therefore y = ce^{\frac{x^3}{3y^3}},\ \textit{which is a general solution.}$$

## Exercise-IV

1. Check the exactness of the following differential equations and solve it.

   1. $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy$. (Ans: $x^5 - 5x^2y^2 + 5y^4x + 5\cos y = c$.)

   2. $(\sin x \cos y + e^x)dx + (\cos(xy)x^2 + e^y)dy = 0$. (Ans: $e^x - \cos x \cos y + \tan y = c$.)

   3. $(xy\cos(xy) + \sin(xy))dx + (\cos x \sin y + \sec^2 y)dy = 0$. (Ans: $x\sin(xy) + e^y = c$.)

   4. $(2xy + y + -\tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$. (Ans: $x^2y + xy - x\tan y + \tan y = c$.)

   5. $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$. (Ans: $e^{xy^2} + x^4 - y^3 = c$.)

   6. $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)ydy = 0$. (Ans: $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$.)

   7. $y\sin 2x dx = (1 + y^2 + \cos^2 x)dy$. (Ans: $3y\cos 2x + 6y + 2y^3 = c$.)

   8. $\frac{2x}{y^3}dx + \frac{y^2-3x^2}{y^4}dy = 0$. (Ans: $x^2 - y^2 = cy^3$.)

   9. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy$. (Ans: $y(x + \log x) + x\cos y = c$.)

   10. $(\sin x \sin y + \sec^2 x)dx + (\tan^2 y - \cos x \cos y)dy = 0$. (Ans: $\tan x - \cos x \sin y + \tan y - y = c$.)

2. Solve the following differential equations using integrating factor.

   1. $(xy\sin(xy) + \cos(xy))ydx + (xy\sin(xy) - \cos(xy))xdy = 0$. (Ans: $x = cy\cos xy$.)

   2. $x^2ydx - (x^3 + y^3)dy = 0$. (Ans: $y = ce^{\frac{x^3}{3y^3}}$.)

   3. $(y + y^2 - y^3)dx - (x + xy^2 - y)dy = 0$. (Ans: $x + xy + y\log y - xy^2 = cy$.)

   4. $ydx + (y - x)dy = 0$. (Ans: $ye^{\frac{x}{y}} = c$.)

   5. $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$. (Ans: $x - 2y\log x + 3y\log y = cy$.)

# Index 3

# Differential Equation of First order and Higher degree.

The general form of differential equation of first order and higher degree is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots + P_{n-1}\frac{dy}{dx} + P_n = 0.$$

Where each $P_i$ is a function of $x$ and $y$. If $\frac{dy}{dx} = p$, then the general form reduces to

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0.$$

Hence it also can be written as $F(x, y, p) = 0$. In this chapter we study following methods of solving differential equation of first order and higher degree.
**Method of solving differential equation of the form** $F(x, y, p) = 0$**.**

1. Differential equations which are solvable for $p$.
2. Differential equations which are solvable for $x$.
3. Differential equations which are solvable for $y$.
4. Clairaut's differential equations.
5. Lagrange's differential equations.

## 3.1 Differential equations which are solvable for $p$.

Suppose we can write the differential equation $F(x,y,p)=0$ of degree $n$ in the form

$$(p-f_1(x,y))(p-f_2(x,y))(p-f_3(x,y))\cdots(p-f_n(x,y))=0. \tag{3.1}$$

Now comparing each factor with zero we get $p-f_i(x,y)=0$, where $i=1,2,\ldots,n$. Which is linear differential equation. Suppose solution of $p-f_i(x,y)=0$ is given by $F_i(x,y,c_i)=0$. Where $c_i$ is an arbitrary constant. Instead of taking different $c_i$'s in the general solution of $p-f_i(x,y)=0$ if we take only one $c$ in all, then it makes no difference in general solution. Therefore general solution $p-f_i(x,y)=0$ will be $F_i(x,y,c)=0$. Then general solution of equation (3.1) is given by $F_1(x,y,c)F_2(x,y,c)\cdots F_n(x,y,c)=0$. Thus, differential equation of n degree and first order having linear factor $p-f_i(x,y)=0$ are known as *solvable for p.*

**Examples 3.1.** *(1)Solve:* $xyp^3+(x^2-2y^2)p^2-2xyp=0$
*Solution: The given differential equation is of degree* 3 *and therefore it has three linear factor.*

$$p[xyp^2+(x^2-2y^2)p-2xy]=0.$$

$$\therefore p[xyp^2+x^2p-2y^2p-2xy]=0.$$

$$\therefore p(xp-2y)(yp+x)=0.$$

*Comparing these three linear factor with zero we get*

1. $p=0 \implies y-c=0.$
2. $xp-2y=0 \implies \frac{dy}{y}=2\frac{dx}{x} \implies y=cx^2.$
3. $yp+x=0 \implies ydy+xdx=0 \implies x^2+y^2-2c=0.$

*Therefore, the general solution is given by multiplying these three solutions of linear factors of given equation.* $\therefore (y-c)(y-cx^2)(x^2+y^2-2c)=0$. *Which is a general solution.*
*(2)Solve:* $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}.$
*Solution: put* $p=\frac{dy}{dx}$ *we get* $p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}.$

$$\therefore p^2+p\left(\frac{y}{x}-\frac{x}{y}\right)-1=0.$$

$$\therefore \left(p+\frac{y}{x}\right)\left(p-\frac{x}{y}\right)=0.$$

*Now comparing the linear factors with zero we get*

1. $\frac{dy}{dx}+\frac{y}{x}=0 \implies xdy+ydx=0. \implies d(xy)=0 \implies xy=c$
2. $\frac{dy}{dx}-\frac{y}{x}=0 \implies xdy-ydx=0. \implies x^2-y^2=c$

*Thus, the general solution can be obtained by multiplying the general solutions of the linear factors of given differential equation.*

$$(xy - c)(x^2 - y^2 - c) = 0.$$

*Which is a general solution.*

**Exercise-V**

Solve the following differential equations.

1. $p^2 - (x+3y)p + 2y(x+y) = 0.$ (Ans. $(y - ce^{-2x})(x + y - 1 - ce^x) = 0.$)
2. $p^2 - 7p + 10.$ (Ans. $(y - 5x - c)(y - 2x - c) = 0.$)
3. $p(p+y) = x(x+y).$ (Ans. $(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0.$)
4. $yp^2 + (x-y)p - x.$ (Ans. $(x - y + c)(x^2 + y^2 + c) = 0.$)
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ (Ans. $(y - c)(y + x^2 - c)(xy + cy + 1) = 0.$)
6. $p^2 + 2py\cot x - y^2 = 0.$ (Ans. $y(1 \pm \cos x) = c.$)
7. $x^2p^2 + xyp - 6y^2 = 0.$ (Ans. $(y - cx^2)(x^3y - c) = 0.$)
8. $y^2p^2 - x^2 = 0.$ (Ans. $(x^2 + y^2 + c)(x^2 - y^2 + c) = 0.$)
9. $p^2 + 2p\cos 2x - \sin^2 x = 0.$ (Ans. $(2y + 2x + \sin 2x + c) = 0.$)

## 3.2 Differential equations which are solvable for $y$.

If the differential equation of the form $F(x, y, p) = 0$ can be written as $y = f(x, p) = 0$, then it is said to be *solvable for y*. In order to solve these types of differential equation we differentiate with respect to $x$ we get

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx} = F\left(x, p, \frac{dp}{dx}\right). \tag{3.2}$$

Which is in variable $p$ and $x$. Hence its solution is given by $g(x, p, c) = 0$. By eliminate $p$ from equation (3.2) and $g(x, p, c)$ we get function $\phi(x, y, c)$ which will be the general solution of the given differential equation. If it is not possible to eliminate $p$, then general solution can be obtained by taking $x = F_1(p, c)$ and $y = F_2(p, c)$. Where $c$ is an arbitrary constant. Let us see following examples to understand this method.

**Examples 3.2.** *(1).Solve:* $xp^2 - 2yp + ax = 0$
*Solution: Here,* $y = \frac{1}{2}xp + \frac{1}{2}\frac{ax}{p}$*; by differentiating with respect to* $x$ *we get*

$$\frac{dy}{dx} = \frac{1}{2}p + \frac{1}{2}x\frac{dp}{dx} + \frac{a}{2p} - \frac{ax}{2p^2}\frac{dp}{dx}.$$

$$\therefore p = \frac{1}{2}p + \left(\frac{1}{2}x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \frac{1}{2}\frac{a}{p}.$$

$$p = \left(x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \frac{a}{p} \implies p^3 - p^2x\frac{dp}{dx} + ax\frac{dp}{dx} - ap = 0.$$

$$\therefore (p^3 - a)\left(p - x\frac{dp}{dx}\right) = 0.$$

$$\therefore p - x\frac{dp}{dx} = 0 \text{ or } p^3 - a = 0.$$

$$\therefore \frac{dp}{p} = \frac{dx}{x} \implies \log p = \log x + \log c.$$

$$\therefore p = cx.$$

*Now, substitute* $p = cx$ *in* $y = \frac{1}{2}xp + \frac{1}{2}\frac{ax}{p}$ *we get,* $y = \frac{1}{2}cx^2 + \frac{1}{2}\frac{a}{c}$. *Which is a general solution.*
*(2)* $xp - y + x^{\frac{3}{2}} = 0$.
*Solution:The given equation can be express in the form* $y = f(x,p)$. *Therefore it is solvable for* $y$. $y = xp + x^{\frac{3}{2}}$. *Differentiate with respect to x we get,*

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{3}{2}x^{\frac{1}{2}}.$$

$$\therefore p = p + x\frac{dp}{dx} + \frac{3}{2}x^{\frac{1}{2}} \implies \frac{dp}{dx} + \frac{3}{\sqrt{x}} = 0.$$

$$\therefore \int dp + \frac{3}{2}\int \frac{dx}{\sqrt{x}} = c \implies p + 3\sqrt{x} = c.$$

$$\therefore p = c - 3\sqrt{x}.$$

*Now to eliminate p, substitute its value in equation* $y = xp + x^{\frac{3}{2}}$ *we get,*

$$y = cx - 2x^{\frac{3}{2}}. \text{ Which is general solution.}$$

*(3) Solve:* $x + 2(xp - y) + p^2 = 0$.
*Solution: The given equation can be express in the form* $y = f(x,p)$. *Therefore it is solvable for* $y$. $y = \frac{1}{2}x + xp + \frac{1}{2}p^2$. *Differentiate with respect to x we get,*

$$\frac{dy}{dx} = p = \frac{1}{2} + p + x\frac{dp}{dx} + p\frac{dp}{dx}.$$

$$\therefore (x+p)\frac{dp}{dx} + \frac{1}{2} = 0.$$

*Now put* $x + p = u$ *we get* $1 + \frac{dp}{dx} = \frac{du}{dx}$.

$$\therefore u\left(\frac{du}{dx} - 1\right) + \frac{1}{2} = 0.$$

$$\therefore \frac{du}{dx} = \frac{2u-1}{2u} \implies \frac{2u}{2u-1}du = dx.$$

$$\therefore \int \left(1 + \frac{1}{2u-1}\right) + \int dx + c.$$

$$\therefore u + \frac{1}{2}\log(2u-1) = x + c.$$

$$\therefore x + p + \frac{1}{2}\log(2x+2p-1) = x + c.$$

$$\therefore 2p + \frac{1}{2}\log(2x+2p-1) = c.$$

$$\therefore 2x + 2p - 1 = e^{2p-c}.$$

$$\therefore x = \frac{1}{2}e^{2p-c} + 1 - p.$$

*Here we can not eliminate p from above equation. Hence, the general solution can be obtained from* $y = \frac{1}{2}x + xp + \frac{1}{2}p^2$ *and* $x = \frac{1}{2}e^{2p-c} + 1 - p$.

## 3.3 Differential equations which are solvable for $x$.

If the differential equation of the form $F(x, y, p) = 0$ can be written as $x = f(y, p) = 0$, then it is said to be *solvable for x*. In order to solve these types of differential equation we differentiate with respect to $y$ we get

$$\frac{dx}{dy} = p = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy} = F\left(x, p, \frac{dp}{dy}\right). \tag{3.3}$$

Which is in variable $p$ and $y$. Hence its solution is given by $g(y, p, c) = 0$. By eliminate $p$ from equation (3.3) and $g(y, p, c)$ we get function $\phi(x, y, c)$ which will be the general solution of the given differential equation. If it is not possible to eliminate $p$, then general solution can be obtained by taking $x = F_1(p, c)$ and $y = F_2(p, c)$. Where $c$ is an arbitrary constant. Let us see following examples to understand this method.

**Examples 3.3.** *(1)Solve:* $y^2p^2 - 3xp + y = 0$.
*Solution: The given differential equation is of the form* $x = f(y, p)$, *where* $f(y, p) = \frac{1}{3}\left(\frac{y}{p} + y^2p\right)$. *Now differentiate with respect to y we get*

$$\therefore 3\frac{dx}{dy} = 3\frac{1}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + 2yp + y^2\frac{dp}{dy}.$$

$$\therefore 2yp - \frac{2}{p} + \left(y^2 - \frac{y}{p^2}\right)\frac{dp}{dy} = 0.$$

$$\therefore 2p(yp^2 - 1) + y(yp^2 - 1)\frac{dp}{dy} = 0.$$

$$\therefore (yp^2 - 1)\left(2p + y\frac{dp}{dy}\right) = 0.$$

*We ignore* $yp^2-1=0$ *we get and consider* $2p+y\frac{dp}{dy}=0$.

$$\therefore \frac{dp}{p}+2\frac{dy}{y}=0.$$

$$\therefore \log p+2\log y=\log c.$$

$$\therefore py^2=c \implies p=\frac{c}{y^2}.$$

*Hence, substitute value of* $p$ *we get* $y^3-2cx+c^2=0$. *Which is a general solution. (2)Solve:*$x=p+\frac{1}{p}$. *Solution: It is easy too see that this differential equation is solvable for* $x$. *By differentiating with respect to* $y$ *we get*

$$\therefore \frac{dx}{dy}=\frac{1}{p}=\frac{dp}{dy}-\frac{1}{p^2}\frac{dp}{dy}.$$

$$\therefore \frac{1}{p}=\left(1-\frac{1}{p^2}\right)\frac{dp}{dy} \implies \left(\frac{p^2-1}{p}\right)dp=dy.$$

$$\therefore \int\left(p-\frac{1}{p}\right)dp=\int dy+c.$$

$$\therefore y=\frac{p^2}{2}-\log p+c.$$

*Where* $c$ *is an arbitrary constant. Here, it is difficult to eliminate* $p$. *Therefore, general solution can be obtained by taking* $x=p+\frac{1}{p}; y=\frac{p^2}{2}-\log p+c$.

**Exercise-VI**

1. $y=(1+p)x+p^2$. (Ans:$x=-2p+2+ce^{-p}; y=2-p^2+c(1+p)e^{-p}$.)
2. $xp-y+\sqrt{x}$. (Ans:$y=cx+2\sqrt{x}$.)
3. $y=2p+3p^2$. (Ans:$x=2p+3p^2; y=2\log p+3p+c$.)
4. $y+px=p^2x^4$. (Ans:$xy=c^2x-c$.)
5. $y^2p^2-3xp+y=0$. (Ans:$y^3-3cx+c^2=0$.)
6. $y=2px-p^2$. (Ans:$x=\frac{2}{3}p+cp^{-2}; y=\frac{1}{3}p^2+\frac{2c}{p}$.)
7. $y^2+p^2=0$. (Ans:$y=\pm\sin(x+c)$.)
8. $p^2y+2px=y$. (Ans:$y^2=2cx+c^2$.)
9. $y-2px=\tan^{-1}p$. (Ans:$2\sqrt{cx}+\tan^{-1}c$.)
10. $xp^2-yp-y=0$. (Ans:$c(1+p)e^p; y=cp^2c^p$.)
11. $y=x+a\tan^{-1}p$. (Ans:$x+c=\frac{a}{2}\left[\log(p-1)-\frac{1}{2}\log(1+p^2)-\tan^{-1}p\right]; y=x+a\tan^{-1}p$.)

12. $x^2 = a^2(1+p^2)$. (Ans:$x = a\sqrt{1+p^2}; y = \frac{a}{2}\left[p\sqrt{1+p^2} - \log(p+\sqrt{p+\sqrt{p^2+1}})\right] + c$.)

13. $p^2 = (p-1)y$. (Ans:$x = \log(p-1) + \frac{1}{p-1} + c; y = \frac{p^2}{p-1}$.)

14. $x = \frac{p}{1+p^2} + \tan^{-1} p$. (Ans:$x = x = \frac{p}{1+p^2} + \tan^{-1} p; y = c - \frac{1}{1+p}$.)

15. $p^2 - 4xyp + 8p^2 = 0$. (Ans:$c(c-4x)^2 = 64y$.)

## 3.4 Clairaut's differential equations.

**Definition 3.4.** *A differential equation of the form $y = px + f(p)$ is known as* Clairaut's differential equation.

It is easy to see that Clairaut's differential equation $y = px + f(p)$ is solvable for $y$. Hence, in order to solve we differentiate with respect to $x$ on both sides we get,

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$$\implies (f'(p) + x)\frac{dp}{dx} = 0.$$

$$\implies \frac{dp}{dx} = 0 \text{ or } x + f'(p) = 0.$$

By taking the case $\frac{dp}{dx} = 0$ we get $p = \frac{dy}{dx} = c$. Where $c$ is an arbitrary constant. Thus, by eliminating $p$ from Clairaut's equation we have the family of straight lines given by $y = cx + f(c)$, as the general solution of Clairaut's differential equation. The later case $x + f'(p) = 0$ defines only one solution $y(x)$ , so-called singular solution, whose graph is the envelope of the graphs of the general solutions. The singular solution is usually represented using parametric notation, as $(x(p), y(p))$, where p represents $\frac{dy}{dx}$.

**Examples 3.5.** *(1) Solve: $x^2(y - px) = yp^2$.*
*Solution: The given differential equation is not Clairaut's differential equation, but by taking $x^2 = u$ and $y^2 = v$ we can convert it into the Clairaut's form. $x^2 = u \implies 2xdx = du$, and $y^2 = v \implies 2ydy = dv$. $\therefore \frac{y}{x}\frac{dy}{dx} = \frac{dv}{du} \implies p = \frac{x}{y}\frac{dv}{du}$. Now given equation reduces to*

$$x^2\left(y - \frac{x^2}{y}\frac{dv}{du}\right) = y\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2.$$

$$y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2.$$

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2.$$

*Which is Clairaut's differential equation. Hence, the general solution can be obtained by taking* $\frac{dv}{du} = c$.
*Hence* $v = cu + c^2$ *and* $y^2 = cx^2 + c^2$ *is the general solution.*
*(2) Solve:* $\sin px \cos y = \cos px \sin y + p$.
*Solution: The given differential equation is not of the Clairaut's form. Notice that,*

$$\sin px \cos y - \cos px \sin y = p \implies \sin(px - y) = p$$

.

$$px - y = \sin^{-1} p.$$

$$y = px + \sin^{-1} p, \textit{which is in Clairaut's form.}$$

$$p = c \implies y = cx + \sin^{-1} c, \textit{ which is a general solution.}$$

*(3)Solve:* $e^{4x}(p-1) + e^{2y}p^2 = 0$.
*Solution: The given differential equation is not of the Clairaut's form, but by taking* $e^{2x} = u$ *and* $e^{2y} = v$ *we can convert it into Clairaut's form.*

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2. \textit{ Which is in Clairaut's form.}$$

$$\frac{dv}{du} = c \implies v = uc + c^2 \implies e^{2y} = ce^{2x} + c^2. \textit{ Which is a general solution.}$$

## 3.5 Lagrange's differential equation.

**Definition 3.6.** *A differential equation of the form* $y = xf(p) + F(p)$ *is known as* Lagrange's differential equation.

It is easy to see that Lagrange's differential equation is solvable for $y$. Hence, in order to solve this differential equation we differentiate with respect to $x$ on both sides we get

$$\frac{dy}{dx} = p = f(p) + xf'(p)\frac{dp}{dx} + F'(p)\frac{dp}{dx}.$$

$$\therefore p - f(p) = [xf'(p) + F'(p)]\frac{dp}{dx}.$$

$$\therefore \frac{dx}{dp} = \frac{xf'(p) + F'(p)}{p - f(p)}.$$

$$\therefore \frac{dx}{dp} = \frac{f'(p)}{p - f(p)}x + \frac{F'(p)}{p - f(p)}.$$

Which is linear in $x$ and $p$. So it can be solved by method of linear differential equation $\frac{dy}{dx} + Py = Q$, where $P$ and $Q$ are functions of $x$ only.

**Remark 3.7.** 1. *An equation of the form* $x = yf(q) + F(q)$, *where* $q = \frac{dx}{dy}$ *is also known as Lagrange's differential equation and also can be solved by using method to solve differential equation which are solvable for* $x$.

2. *By taking $f(p) = p$ in Lagrange's differential equation we get Clairaut's differential equation. Thus, Clairaut's differential equation is a particular case of Lagrange's differential equation.*

**Examples 3.8.** *(1)Solve: $y = 2px - \frac{1}{3}p^2$.*
*Solution: The given differential equation is solvable for $y$. In order to solve we differentiate with respect to x on both sides we get,*

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} - \frac{2}{3}p\frac{dp}{dx}.$$
$$\therefore -p = 2\left(x - \frac{1}{3}p\right)\frac{dp}{dx} \implies p\frac{dx}{dp} + 2\left(x - \frac{1}{3}p\right) = 0.$$
$$\therefore \frac{dx}{dp} + \frac{2}{p}x = \frac{2}{3}.$$

*Which is linear in variables x and $p$. Thus, solution can be obtained by,*

$$x(I.F.) = \int Q(I.F.)dx + c$$

*Where $I.F. = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$ and $Q = \frac{2}{3}$.*

$$\therefore xp^2 = \int \frac{2}{3}p^2 dp + c = \frac{2}{3}\frac{p^3}{3} + c. \text{ Where } c \text{ is an arbitrary constant.}$$

$$\therefore x = \frac{2}{9}p + \frac{c}{p^2}.$$

*Substitute this value of $x$ in given equation we get $y = \frac{1}{9}p^2 + \frac{2c}{p}$. Hence, $x = \frac{2}{9}p + \frac{c}{p^2}$ and $y = \frac{1}{9}p^2 + \frac{2c}{p}$ is a general solution.*

**Exercise:VII**

Solve the following differential equation.

1. $y = px + p - p^2$. (Ans:$y = cx + c - c^2$.)
2. $y = px + \frac{m}{p}$.(Ans:$y = cx + \frac{m}{c}$.)
3. $y = xp - p^2 + \log p$.(Ans:$y = cx - c^2 + \log c$.)
4. $(x-a)p^2 + (x-y)p - y = 0$.(Ans:$y = cx - a\frac{c^2}{c+1}$.)
5. $y^2p^3 - 2xp + y = 0$.(Ans:$y^2 = cx - \frac{1}{8}c^3$.)
6. $x + yp = a + bp$.(Ans:$x^2 + y^2 = 2(ax + by + c)$.)
7. $p^2 - 6px + 3y = 0$. (Ans:$x = \frac{2p}{9} + \frac{3c}{p^2}$; $y = \frac{p^2}{9} + \frac{6c}{p}$.)
8. $x + y = \left(\frac{1+p}{1-p}\right)^2$. (Ans:$x = \frac{2}{(1-p)^2} + k$; $y = \frac{p^2+2p-1}{(1-p)^2} - k$.)

9. $p^2 = (p-1)y$(Ans:$x = \log(p-1) + \frac{1}{p-1} + c; y = \frac{p^2}{p-1}$.)

10. $e^{3x}(p-1) + p^3 e^{2y} = 0$.(Ans:$e^y = ce^x + c^3$.)

11. $(px-y)(x-py) = 2p$.(Hint: $x^2 = u, y^2 = v$). (Ans:$c^2x^2 - c(x^2+y^2-2) + y^2 = 0$.)

12. $p^3 - xp - y = 0$.(Ans:$x = \frac{3}{5}p^2 + \frac{k}{\sqrt{p}}$ and $y = \frac{2}{5}p^3 - k\sqrt{p}$.)

13. $p^2(x-5) + (2x-y)p - 2y = 0$.(Ans:$y = cx - \frac{5c^2}{c+4}$.)

14. $p^2 + 2p\cos 2x - \sin^2 2x = 0$.(Ans:$(2y+2x+\sin 2x + c)(2y - 2x + \sin 2x + c) = 0$.)

15. $y^2 = xyp + \frac{y^3p^3}{x^3}$.(Hint: $x^2 = u, y^2 = v$).(Ans:$y^2 = cx^2 + c^3$.)

16. $y^2(y-xp) = x^4p^2$. (Hint: $x = \frac{1}{u}, y = \frac{1}{v}$).(Ans:$\frac{1}{y} = \frac{c}{x} + c^2$.)

# Index 4

# Higher Order Linear Differential Equation

**Definition 4.1.** *If $P_1, P_2, \ldots, P_n, X$ are functions of x or constants, then*

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = X \tag{4.1}$$

*is called $n^{th}$* order linear differential equation.

In equation (4.1) if $X = 0$, then equation is called homogeneous linear differential equation, otherwise it said to be non-homogeneous differential equation.

Solution of linear equation (4.1) can be separated into two parts.
(a) $P_1, P_2, \ldots, P_n$ are constants.
(b) $P_1, P_2, \ldots, P_n$ are functions of $x$.

In this chapter we discuss the different methods to solve linear differential equation of type (a).

**Theorem 4.2.** *If $y_1$ and $y_2$ are solutions of equation*

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = 0 \tag{4.2}$$

*then $c_1 y_1 + c_2 y_2$ $(= u)$ is also its solution, where $c_1$ and $c_2$ are arbitrary constants.*

*Proof.* Since $y = y_1$ and $y = y_2$ are solution of (4.2),

$$\frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots P_n y_1 = 0 \tag{4.3}$$

$$\frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots P_n y_2 = 0 \tag{4.4}$$

Then
$$\begin{aligned}
&\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + P_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \cdots P_n (c_1 y_1 + c_2 y_2) \\
&= c_1 \left( \frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots P_n y_1 \right) + c_2 \left( \frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots P_n y_2 \right) \\
&= c_1(0) + c_2(0) = 0 \qquad \text{[by (4.3) and (4.4)]}
\end{aligned}$$

$$i.e \quad \frac{d^n u}{dx^n} + P_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots P_n u = 0 \tag{4.5}$$

This proves the theorem. □

Since the general solution of $n^{\text{th}}$ order differential equation contains $n$ arbitrary constants, it follows, from the above, that if $y_1, y_2, ..., y_n$ are n solution of (4.2), then $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ $(= u)$ is a solution of (4.2). This solution is called the **Complementary function (C.F.)** of equation (4.2).
If we denote the complementary
Suppose that $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots k_n y = X \tag{4.6}$$

where $k_1, k_2, ... k_n$ are arbitrary constants.

$$\text{Then} \qquad \frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots k_n v = X \tag{4.7}$$

Adding (4.5) and (4.7), we have $\dfrac{d^n (u+v)}{dx^n} + k_1 \dfrac{d^{n-1} (u+v)}{dx^{n-1}} + \cdots k_n (u+v) = X$
This shows that $y = u + v$ us the complete solution of (4.6). Here $y = v$ is called the ***Particular solution(P.I.)*** of (4.6).

$\therefore$ The general solution (**G.S.**) of (4.6) is $y =$ **C.F.** + **P.I.**

Thus in order to solve the equation (4.6), we have to first find the *C. F.* , and then the *P. I.* . For a homogeneous differential equation the *C. F.* and *G. S.* will be same.

## 4.1 Operator '$D$ '

To find the solution of linear differential equation, operator '$D$ ' play very important role. '$D$ ' is defined as follow

$$D = \frac{d}{dx},\ D^2 = \frac{d^2}{dx^2},\ ...,\ D^n = \frac{d^n}{dx^n}$$

$$\therefore \frac{dy}{dx} = Dy;\ \frac{d^2y}{dx^2} = D^2y,\ ...,\frac{d^ny}{dx^n} = D^ny$$

With this notation the equation (4.1) can be written as

$$(D^n + P_1D^{n-1} + \cdots + P_n)y = X \quad i.e.\ f(D)y = X$$

where $f(D) = D^n + P_1D^{n-1} + \cdots + P_n,\ i.e.$ a polynomial in $D$.

Thus the symbol $D$ stands for the operation of differentiation and can be treated much the same as an algebraic quantity *i.e.* $f(D)$ can be factorized by ordinary rules of algebra and the factors may be taken in any order.

## 4.2 Rule to find the Complementary function:

Consider the equation

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots k_ny = 0 \tag{4.8}$$

where $k_1, k_2, ...k_n$ are arbitrary constants.
Then this equation in symbolic form is $(D^n + k_1D^{n-1} + \cdots + k_n)y = X$. Its symbolic co-efficient equated to zero i.e.

$$D^n + k_1D^{n-1} + \cdots + k_n = 0$$

is called the *Auxiliary Equation (A.E.)*.

Since it is an $n^{\text{th}}$ order polynomial equation in terms of $D$, it has $n$ roots say $m_1, m_2, ..., m_n$.

***Case*** : I If all the roots be real and different, then the *G. S.* of (4.8) is given by

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

***Case*** : II If two roots are equal ($i.e.\ m_1 = m_2$), then the *G. S.* of (4.8) is given by

$$y = (c_1 + c_2x)e^{m_1x} + \cdots + c_ne^{m_nx}$$

If, however, the $A.E.$ has three equal roots ($i.e.\ m_1 = m_2 = m_3$), then the *G. S.* of (4.8) is given by

$$y = (c_1 + c_2x + c_3x^2)e^{m_1x} + \cdots + c_ne^{m_nx}$$

***Case*** : III If one pair of roots be imaginary, $i.e.\ m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$, then the *G. S.* of (4.8) is given by

$$y = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x)) + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$$

***Case*** : IV  If two pairs of imaginary roots be equal *i.e.* $m_1 = m_2 = \alpha + i\beta,\ m_3 = m_4 = \alpha - i\beta$, then the *G. S.* of (4.8) is given by

$$y = e^{\alpha x}\big((c_1 + c_2 x)\cos(\beta x) + (c_3 + c_4 x)\sin(\beta x)\big) + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$$

**Example 4.3.** *Solve* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0.$

**Sol.** Let $D = \frac{d}{dx}$. Then given equation reduces to $(D^2 + D - 2)y = 0$.
Its A.E. is $D^2 + D - 2 = 0,\ i.e.\ \ (D+2)(D-1) = 0$ whence $D = -2, 1$.
Hence the *G. S.* is $y = c_1 e^{-2x} + c_2 e^{1x}$.

**Example 4.4.** *Solve* $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0.$

**Sol.** Let $D = \frac{d}{dx}$. Then given equation reduces to $(D^2 + 6D + 9)y = 0$.
Its A.E. is $D^2 + 6D + 9 = 0,\ i.e.\ \ (D+3)^2 = 0$ whence $D = -3, -3$.
Hence the *G. S.* is $y = (c_1 + c_2 x)e^{-3x}$.

**Example 4.5.** *Solve* $(D^3 + D^2 + 4D + 4)y = 0.$

**Sol.** Here the *A.E.* is $D^3 + D^2 + 4D + 4 = 0 \quad i.e.(D^2+4)(D+1) = 0 \quad \therefore\ D = -1, \pm 2i.$
Hence the *G. S.* is $y = c_1 e^{-x} + e^{0x}[c_2\cos(2x) + c_3\sin(2x)] = c_1 e^{-x} + c_2\cos(2x) + c_3\sin(2x)$

**Example 4.6.** *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

**Sol.** Let $D = \frac{d}{dt}$. Then given equation reduces to $(D^4 + 4)x = 0$.
Its A.E. is $D^4 + 4 = 0$.

$$\begin{aligned}
&\therefore\ D^4 + 4D^2 + 4 - 4D^2 = 0\\
&\therefore\ (D^2+2)^2 - (2D)^2 = 0\\
&\therefore\ (D^2 + 2 + 2D)(D^2 + 2 - 2D) = 0\\
&\therefore\ D^2 + 2 + 2D = 0 \quad \text{or} \quad D^2 + 2 - 2D = 0\\
&\therefore\ D = \frac{-2 \pm \sqrt{-4}}{2} \quad \text{or} \quad \frac{2 \pm \sqrt{-4}}{2}\\
&\therefore\ D = -1 \pm i \quad \text{or} \quad D = 1 \pm i
\end{aligned}$$

Thus the *G. S.* is $y = e^{-t}[c_1\cos(t) + c_2\sin(t)] + e^{t}[c_3\cos(t) + c_4\sin(t)]$.

**Exercise-I**

Que :1  Solve the following differential equation.

1. $y'' - 2y' + 10y = 0$ Ans. $y = e^x [c_1 \cos(3x) + c_2 \sin(3x)]$
2. $4y''' + 4y'' + 4y' = 0$ Ans. $y = c_1 + (c_2 + c_3 x)\, e^{\frac{-x}{2}}$
3. $\dfrac{d^3y}{dx^3} + y = 0$ Ans. $y = c_1 e^{-x} + e^{\frac{x}{2}} \left( c_2 \cos\left(\frac{\sqrt{3}}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\right)\right)$
4. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$ Ans. $y = (c_1 + c_2 x + c_3 x^2)e^x$
5. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$ Ans. $y = (c_1 + c_2 x)\cos(2x) + (c_3 + c_4 x)\sin(2x)$
6. $\dfrac{d^4y}{dx^4} + a^4 y = 0$ Ans $y = e^{\frac{a}{\sqrt{2}}x}\left(c_1 \cos\left(\frac{a}{\sqrt{2}}\right) + c_2 \sin\left(\frac{a}{\sqrt{2}}\right)\right) + e^{-\frac{a}{\sqrt{2}}x}\left(c_3 \cos\left(\frac{a}{\sqrt{2}}\right) + c_4 \sin\left(\frac{a}{\sqrt{2}}\right)\right)$.

Que : 2 If $\dfrac{d^4x}{dt^4} = m^4 y$, show that $x = c_1 \cos(mt) + c_2 \sin(mt) + c_3 \cosh(mt) + c_4 \sinh(mt)$.

$\left(\text{Hint: Use } \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}\right)$

## 4.3 Inverse Operator:

1. **Definition:** $\dfrac{1}{f(D)}X$ is that function of $x$, not containing arbitrary constants which when operated upon by $f(D)$ gives $X$.

$$i.e. \quad f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $y = \dfrac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and is, therefore, its *particular integral.*

2. $\mathbf{\dfrac{1}{D}X = \int X\, dx}$.

Let $\dfrac{1}{D}X = y$.

Operating by $D$, $D\dfrac{1}{D}X = Dy$. $i.e.$ $X = \frac{dy}{dx}$.

Integrating both the sides w.r.t. $x$, we get $y = \int X dx$.

Thus $\dfrac{1}{D}X = \int X dx$.

3. $\mathbf{\dfrac{1}{D-a}X = e^{ax} \int Xe^{-ax}\, dx}$.

Let $\dfrac{1}{D-a}X = y$.

Operating by $D - a$, $(D-a)\dfrac{1}{D-a}X = (D-a)y$. $\Rightarrow X = \frac{dy}{dx} - ay$.

$i.e.$ $\dfrac{dy}{dx} - ay = X$, which is a linear equation in first order.

So solution is $ye^{-ax} = \int Xe^{-ax}dx$ $\Rightarrow y = e^{ax}\int Xe^{-ax}dx$.

Thus $\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$

**Example 4.7.** *Find* $\frac{1}{D^2+2D-15}e^{2x}$.

**Sol.**
$$\begin{aligned}\frac{1}{D^2+2D-15}e^{2x} &= \frac{1}{(D+5)(D-3)}e^{2x}\\ &= \frac{1}{(D+5)}\frac{1}{(D-3)}e^{2x}\\ &= \frac{1}{(D+5)}e^{3x}\int e^{-3x}e^{2x}dx \qquad \left(\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.\right)\\ &= -\frac{1}{(D+5)}e^{2x}\\ &= -e^{-5x}\int e^{5x}e^{2x}dx\\ &= -\frac{1}{7}e^{2x}\end{aligned}$$

## 4.4 Rules for finding the *Particular Integral*

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots k_n y = X$,
which in symbolic form is $(D^n + k_1 D^{n-1} + \cdots + k_n)y = f(D)y = X$.

$$\therefore P.I. = \frac{1}{D^n + k_1 D^{n-1} + \cdots + k_n}X = \frac{1}{f(D)}X$$

*Case* : I **When X = e^ax^.**

If $f(a) \neq 0$, then $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$.

If $f(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$, provided $f'(a) \neq 0$.

If $f(a) = 0$ and $f'(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$, and so on.

*Case* : II **When X = sin (ax + b) or cos (ax + b).**

If $f(-a^2) \neq 0$, then $\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$.

If $f(-a^2) = 0$, then $\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)$, provided $f'(-a^2) \neq 0$.

If $f(a) = 0$ and $f'(a) = 0$, then $\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$, provided $f''(-a^2) \neq 0$, and so on.

Similarly if $f(-a^2) \neq 0$, then $\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$.

If $f(-a^2) = 0$, then $\frac{1}{f(D^2)}\cos(ax+b) = x\frac{1}{f'(-a^2)}\cos(ax+b)$, provided $f'(-a^2) \neq 0$.

If $f(a) = 0$ and $f'(a) = 0$, then $\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\dfrac{1}{f''(-a^2)}\cos(ax+b)$, provided $f''(-a^2) \neq 0$, and so on.

**Example 4.8.** *Solve* $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$

**Sol.** Here given differential equation is non-homogeneous. So general solution is $y = C.F. + P.I.$

To find $C.F$ consider the equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\dfrac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$

Let $D = \dfrac{d}{dx}$. Then this equation reduces to $(D^2 - 5D + 6)y = 0.$

And A.E. is $D^2 - 5D + 6 = 0. \Rightarrow (D-3)(D-2) = 0. \Rightarrow D = 3, 2.$

Thus $C.F. = c_1e^{3x} + c_2e^{2x}.$

And

$$\begin{aligned} P.I. &= \frac{1}{D^2-5D+6}e^{4x} \\ &= \frac{1}{16-20+6}e^{4x} \qquad \left(\because \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}\right) \\ &= \frac{1}{2}e^{4x} \end{aligned}$$

Now $G.S. = C.F. + P.I.\,.$

$$\Rightarrow y = c_1e^{3x} + c_2e^{2x} + \frac{1}{2}e^{4x}$$

**Example 4.9.** *Solve* $6\dfrac{d^2y}{dx^2} + 17\dfrac{dy}{dx} - 14 = \sin(3x).$

**Sol.** Here given differential equation is non-homogeneous. So general solution is $y = C.F. + P.I.$

To find $C.F$ consider the equation $6\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 25\dfrac{\mathrm{d}y}{\mathrm{d}x} + 14 = 0.$

Let $D = \dfrac{d}{dx}$. Then this equation reduces to $(6D^2 + 25D + 14)y = 0.$

And A.E. is $6D^2 + 25D + 14 = 0. \Rightarrow (3D+2)(2D+7) = 0. \Rightarrow D = -\tfrac{2}{3}, -\tfrac{7}{2}.$

Thus $C.F. = c_1e^{-\frac{2}{3}x} + c_2e^{-\frac{7}{2}x}.$

And

$$\begin{aligned} P.I. &= \frac{1}{6D^2+25D+14}\sin(3x) \\ &= \frac{1}{6(-9)+25D+14}\sin(3x) \qquad \left(\because \frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)\right) \\ &= \frac{1}{5}\cdot\frac{1}{(5D-6)}\cdot\frac{(5D+6)}{(5D+6)}\sin(3x) \\ &= \frac{1}{5}\cdot\frac{(5D+6)}{(-45+6)}\sin(3x) \\ &= -\tfrac{1}{405}[5D\sin(3x) + 6\sin(3x)] \\ &= -\tfrac{1}{405}[15\cos(3x) + 6\sin(3x)] \end{aligned}$$

Now $\quad G.\ S. = C.\ F. + P.\ I.\ .$

$$\Rightarrow y = c_1 e^{-\frac{2}{3}x} + c_2 e^{-\frac{7}{2}x} - \tfrac{1}{405}[15\cos(3x) + 6\sin(3x)]$$

**Exercise-II**

Que : 1 Find the value of $(i)\ \dfrac{1}{D^2+2D-15}e^{2x}$. $(ii)\ \dfrac{1}{D^2+6D-9}e^{-3x}$. $(iii)\ \dfrac{1}{D^3+D^2-D-1}\cos(2x)$.

$\left(Ans.:\ (i) - \frac{1}{7}e^{2x}\ \ (ii)\frac{1}{2}x^2e^{-3x}\ \ (iii) - \frac{1}{25}(2\sin(2x)+\cos(2x))\right)$

Que : 2 Solve the following differential equation.

1. $(D^3 - 6D^2 + 11d - 6)y = e^{-2x}e^{-3x}$ Ans. $y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{120}(2e^{-2x} + e^{-3x})$
2. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh(x)$ Ans. $y = e^{-2x}[c_1\cos(x) + c_2\sin(x)] - \frac{e^x}{10} - \frac{e^{-x}}{2}$
3. $(D+1)(D-3)^2y = e^{3x} + e^{5x}$ Ans. $y = (c_1 + c_2x)e^{3x} + c_3e^{-x} + \frac{1}{8}x^2e^{3x} + \frac{1}{24}e^{5x}$
4. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 3x = \sin(t)$ Ans. $y = e^{-t}\left[c_1\cos\left(\sqrt{2}t\right)c_2\sin\left(\sqrt{2}t\right)\right] + \frac{1}{4}[\sin(t) - \cos(t)]$
5. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = \cos(5x+3)$ Ans. $y = c_1e^x + c_2e^{3x} - \frac{1}{442}[10\sin(5x+3) + 11\cos(5x+3)]$
6. $(D^2 + 3D + 2)y = \sin(3x)\cos(2x)$ Ans. $y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{884}[10\cos(5x) - 11\sin(5x)] + \frac{1}{20}[\sin(x) + 2\cos(x)]$
7. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{-x} + \sin(2x)$ Ans. $y = c_1 + (c_2 + c_3x)e^{-x} - \frac{x^2}{2}e^{-x} + \frac{3}{50}\cos(2x) - \frac{2}{25}\sin(2x)$

***Case*** : III **When $\mathbf{X = x^m}$.**

Here $P.\ I. = \dfrac{1}{f(D)}x^m = [f(D)]^{-1}x^m$.

Expand $[f(D)]^{-1}$ in ascending power of D as far as the term in $D^m$ and operate on $x^m$ by term. Since the $(m+1)^{\text{th}}$ and higher derivatives of $x^m$ are zero, we need not consider terms beyond $D^m$.

**Note:** Use the following formulae to expand $[f(D)]^{-1}$.

(1) $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

(2) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots + (1+m)D^m + \cdots$

(3) $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots + \sum mD^m + \cdots$

(4) $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 - D^5 + \cdots$

***Case*** : IV **When $\mathbf{X = e^{ax}V}$**, where $V$ is a function of $x$.

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$$

***Case*** : V **When X is any other function of x**.

Here $P.\ I. = \dfrac{1}{f(D)}X$.

If $f(D) = (D - m_1)(D - m_2)...(D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots + \frac{A_n}{D - m_n}.$$

$$\begin{aligned}
\therefore\ P.I. &= \frac{1}{f(D)} X. \\
&= \left[ \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots + \frac{A_n}{D - m_n} \right] X. \\
&= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \cdots + A_n \frac{1}{D - m_n} X. \\
&= A_1 e^{m_1 x} \int X e^{-m_1 x}\, dx + A_2 e^{m_2 x} \int X e^{-m_2 x}\, dx + \cdots + A_n e^{m_n x} \int X e^{-m_n x}\, dx.
\end{aligned}$$

$$\left( \because \frac{1}{D - a} X = e^{ax} \int X e^{-ax}\, dx. \right)$$

This method is a general on and therefor can be applicable to obtain a particular integral in any given case.

**Example 4.10.** *Solve* $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4$

**Sol.** Here given differential equation is non-homogeneous. So general solution is $y = C.F. + P.I.$
To find $C.F.$ consider the equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$. Then A.E. $D^2 + D = 0$. $\therefore\ D(D + 1) = 0 \Rightarrow D = 0, -1$.

$$\therefore\ C.F. = c_1 + c_2 e^{-x}$$

$$\begin{aligned}
\text{And}\ \ P.I. &= \frac{1}{D(D + 1)} (x^2 + 2x + 4) \\
&= \frac{1}{D} (D + 1)^{-1} (x^2 + 2x + 4) \\
&= \frac{1}{D} \left(1 - D + D^2 - D^3 + D^4 - \cdots \right) (x^2 + 2x + 4) \\
&= \left( \frac{1}{D} - 1 + D - D^2 + D^3 - \cdots \right) (x^2 + 2x + 4) \\
&= \frac{1}{D} (x^2 + 2x + 4) - (x^2 + 2x + 4) + D(x^2 + 2x + 4) - D^2 (x^2 + 2x + 4) + D^3 (x^2 + 2x + 4) + \cdots \\
&= \int (x^2 + 2x + 4) dx - (x^2 + 2x + 4) + (2x + 2 + 0) - (2 + 0 + 0) + 0 \\
&= \frac{x^3}{3} + x^2 + 4x - x^2 - 2x - 4 + 2x + 2 - 2 \\
&= \frac{x^3}{3} + 4x - 4
\end{aligned}$$

Thus $G.S. = C.F. + P.I. \quad \Rightarrow\ y = c_1 + c_2 e^{-x} + \dfrac{x^3}{3} + 4x - 4$

**Example 4.11.** *Find* P. I. *of* $(D^2-4D+3)\,y = e^{4x}\sin(2x)$

**Sol.**

$$\begin{aligned}
P.\,I. &= \frac{1}{D^2-4D+3}e^{4x}\sin(2x)\\
&= e^{4x}\frac{1}{(D+4)^2-4(D+4)+3}e^{4x}\sin(2x)\\
&= e^{4x}\frac{1}{D^2+4D+3}\sin(2x)\\
&= e^{4x}\frac{1}{(-4)+4D+3}\sin(2x)\\
&= e^{4x}\frac{1}{4D-1}\sin(2x)\\
&= e^{4x}\frac{4D+1}{16D^2-1}\sin(2x)\\
&= e^{4x}\frac{4D+1}{-65}\sin(2x)\\
&= -\frac{e^{4x}}{65}[4D\sin(2x)+\sin(2x)]\\
&= -\frac{e^{4x}}{65}[8\cos(2x)+\sin(2x)]
\end{aligned}$$

**Example 4.12.** *Solve* $(D^2+16)\,y = \tan(4x)$

**Sol.** Here given differential equation is non-homogeneous. So general solution is $y = C.\,F. + P.\,I.$
To find $C.\,F$ consider the equation $(D^2+16)\,y = 0$. Then A.E. $D^2+16=0 \quad \therefore \; D = \pm\,4i$

$$\therefore\ C.F. = c_1\cos(4x) + c_2\sin(4x)$$

$$\begin{aligned}
\text{And }\ P.I. &= \frac{1}{D^2+16}\tan(4x)\\
&= \frac{1}{8i}\left[\frac{1}{D-4i} - \frac{1}{D+4i}\right]\tan(4x)\\
&= \frac{1}{8i}\left[\frac{1}{D-4i}\tan(4x) - \frac{1}{D+4i}\tan(4x)\right]\\
&= \frac{1}{8i}\left[e^{4ix}\int e^{-4ix}\tan(4x)dx - e^{-4ix}\int e^{4ix}\tan(4x)dx\right] \qquad \left(\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.\right)\\
&= \frac{1}{8i}\left[e^{4ix}\int[\cos(4x) - i\sin(4x)]\tan(4x)dx - e^{-4ix}\int[\cos(4x)+i\sin(4x)]\tan(4x)dx\right]\\
&= \frac{1}{8i}\left[e^{4ix}\int[\cos(4x)\tan(4x) - i\sin(4x)\tan(4x)]dx\right.\\
&\qquad\qquad \left. -e^{-4ix}\int[\cos(4x)\tan(4x)+i\sin(4x)\tan(4x)]dx\right]\\
&= \frac{1}{8i}\left[e^{4ix}\int[\sin(4x) - i\sin^2(4x)\cos(4x)]dx\right.\\
&\qquad\qquad \left. -e^{-4ix}\int[\sin(4x)+i\sin^2(4x)\cos(4x)]dx\right]\\
&= \frac{1}{8i}\left[e^{4ix}\left(\int\sin(4x)dx - i\int\sin^2(4x)\cos(4x)dx\right)\right.\\
&\qquad\qquad \left. -e^{-4ix}\left(\int\sin(4x)dx + i\int\sin^2(4x)\cos(4x)dx\right)\right]\\
&= \frac{1}{8i}\left[e^{4ix}\left(-\frac{\cos(4x)}{4} + i\frac{\sin^3(4x)}{12}\right)\right.\\
&\qquad\qquad \left. -e^{-4ix}\left(-\frac{\cos(4x)}{4} - i\frac{\sin^3(4x)}{12}\right)\right] \qquad \left(\because \int f^n(x)f'(x)dx = \frac{f^{n+1}(x)}{n+1}\right)\\
&= \frac{1}{8i}\left[\frac{\sin^3(4x)}{12}\left(e^{4ix}+e^{-4ix}\right) - \frac{\cos(4x)}{4}\left(e^{4ix}-e^{4ix}\right)\right]
\end{aligned}$$

$$Thus\,G.S. = C.F. + P.I. \quad\Rightarrow\ y = c_1\cos(4x) + c_2\sin(4x) + \frac{1}{8i}\left[\frac{\sin^3(4x)}{12}\left(e^{4ix}+e^{-4ix}\right) - \frac{\cos(4x)}{4}\left(e^{4ix}-e^{4ix}\right)\right]$$

**Exercise-III**

Que : 1 Find $(i)\ \frac{1}{D-2}x^3$ and $(ii)\ \frac{1}{D^2-2D+1}x^2e^{3x}$ $\left(Ans.(i)\ -\frac{1}{2}\left(x^3+\frac{3x^2}{2}+\frac{3x}{2}+\frac{3}{4}\right)\ \ (ii)\,\frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{2}\right)\right)$

Que : 2 Solve the following differential equation.

| | |
|---|---|
| 1. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=2x+x^2$ | Ans. $y=(c_1+c_2x)e^{-x}+x^2-2x+2$ |
| 2. $(D^2-6D+9)y=e^{3x}(1+x)$ | Ans. $y=(c_1+c_2x)e^{3x}+e^{3x}\left(\frac{x^2}{2}+\frac{x^3}{6}\right)$ |
| 3. $\dfrac{d^2y}{dx^2}-y=x^2-1$ | Ans. $y=c_1e^x+c_2e^{-x}-1-x^2$ |
| 4. $(6D^2-D-2)y=xe^{-x}$ | Ans. $y=c_1e^{\frac{2x}{3}}+c_2e^{-\frac{x}{2}}$ |
| 5. $(D^2-2D+3)y=\cos(x)+x^2$ | Ans. $y=e^x\left[c_1\cos\left(\sqrt{2}x\right)+c_2\sin\left(\sqrt{2}x\right)\right]$ $+\frac{1}{12}\left[2\cos(x)-3\sin(x)+4x^2+\frac{16x}{3}+\frac{8}{9}\right]$ |
| 6. $(D^3-D)y=2x+1+4\cos(x)+2e^x$ | Ans. $y=c_1+c_2e^x+c_3e^{-x}+xe^x-(x^2+x)-2\sin(x)$ |
| 7. $(D^4-1)y=e^x\cos(x)$ | Ans. $y=c_1e^x+c_2e^{-x}+c_3\cos(x)+c_4\sin(x)-\dfrac{e^x}{5}\cos(x)$ |
| 8. $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=xe^{3x}+\sin(2x)$ | Ans. $y=c_1e^x+c_2e^{2x}+\dfrac{e^{3x}}{4}(2x-3)+\frac{3}{20}\cos(2x)-\frac{1}{20}\sin(2x)$ |
| 9. $\dfrac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x\cos(2x)$ | Ans. $y=c_1\cos\left(\sqrt{2}x\right)+c_2\sin\left(\sqrt{2}x\right)+\dfrac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$ $+\dfrac{e^x}{17}(4\sin(x)-\cos(x))$ |
| 10. $(D^3+2D^2+D)y=x^2e^{2x}+\sin^2(x)$ | Ans. $y=c_1+(c_2+c_3x)e^{-x}+\dfrac{e^{2x}}{18}\left(x^2-\frac{7x}{8}+\frac{11}{6}\right)$ $+\frac{1}{100}(3\sin(2x)+4\cos(2x))$ |
| 11. $(D^2-1)y=x\sin(x)+(1+x^2)e^x$ | Ans. $y=c_1e^x+c_2e^{-x}+\dfrac{xe^x}{12}\left(2x^2-3x+9\right)-\frac{1}{2}(x\sin(x)+\cos(x))$ |

Now we shall study two such forms of linear differential equation with variable co-efficient which can be reduced to linear differential equations with constat co-efficient by suitable substitutions.

## 4.5 Cauchy's homogenous linear equation

An equation of the form

$$x^n\frac{d^ny}{dx^n}+k_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+k_{n-1}x\frac{dy}{dx}+k_ny=X \tag{4.9}$$

where $k'$s are constants and $X$ is a function of $x$, is called *Cauchy's*[1] *homogeneous linear equation.*

Such equation can be reduced to linear differential equation with constant coefficients, by putting

$$x=e^t \text{ or } t=\log x. \text{ Then if } D=\frac{d}{dt}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{1}{x}\quad i.e.\quad x\frac{dy}{dx}=Dy.$$

[1] A French mathematician *Augustin-Louis Cauchy* (1789-1857) who is considered as the father of modern analysis and creator of complex analysis. He published nearly 800 reserch paper of basic importance. Cauchy is also well known for his contribution to differential equation, in finite series, optics and elasticity.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

$i.e.$ $x^2\frac{d^2y}{dx^2} = D(D-1)y$. Similarly, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

After making these substitution in (4.9), that results a linear equation with constat coefficients, which can be solved as before.

**Example 4.13.** *Solve* $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

**Sol.** This is a *Cauchy's homogeneous linear equation.*
Put $x = e^t$, $i.e.$ $t = \log x$, so that $x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dt}$
Then given equation becomes

$$[D(D-1) - D + 1]y = t \text{ or } (D-1)^2 y = t \tag{4.10}$$

which is a linear equation with constant coefficients.
Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.
$\therefore C.F. = (c_1 + c_2 t)e^t$.
And $P.I. = \frac{1}{(D-1)^2}t = (1-D)^{-2}t = (1 + 2D + 3D^2 + \cdots)t = t + 2$.
Hence the solution of (4.10) is $y = (c_1 + c_2 t)e^t + t + 2$.
Put $t = \log x$ or $e^t = x$, we get
$y = (c_1 + c_2 \log x)x + \log x + 2$ as the required solution of given equation.

**Example 4.14.** *Solve* $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

**Sol.** This is a *Cauchy's homogeneous linear equation.*
Put $x = e^t$, $i.e.$ $t = \log x$, so that $x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dt}$
Then given equation becomes

$$[D(D-1) + 4D + 2]y = e^{e^t} \text{ or } (D^2 + 3D + 2)y = e^{e^t} \tag{4.11}$$

which is a linear equation with constant coefficients.
Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.
$\therefore C.F. = c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$.

$$\begin{aligned}\text{And}\quad P.I. &= \frac{1}{(D^2+3D+2)}e^{e^t} = \frac{1}{(D+1)(D+2)}e^{e^t} = \left[\frac{1}{(D+1)} - \frac{1}{(D+2)}\right]e^{e^t} \\ &= \left[\frac{1}{(D+1)}e^{e^t} - \frac{1}{(D+2)}e^{e^t}\right] \\ &= \left[e^{-t}\int e^t e^{e^t}\,dt - e^{-2t}\int e^{2t}e^{e^t}\,dt\right] \qquad \left(\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx.\right)\end{aligned}$$

$$\begin{aligned}
&= x^{-1}\int e^x dx - x^{-2}\int e^x x dx \qquad (\because\ e^t = x)\\
&= x^{-1}e^x - x^{-2}(xe^x - e^x)\\
&= x^{-2}e^x
\end{aligned}$$

Hence the required solution of $y = c_1 x^{-1} + c_2 x^{-2} + x^{-2}e^x$.

## 4.6 Legendre's linear equation

An equation of the form

$$(ax+b)^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + k_1(ax+b)^{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + k_{n-1}(ax+b)\frac{\mathrm{d}y}{\mathrm{d}x} + k_n y = X \tag{4.12}$$

where $k'$s are constants and $X$ is a function of $x$, is called *Legendre's*[2] *homogeneous linear equation.*

Such equation can be reduced to linear differential equation with constant coefficients, by putting

$$ax + b = e^t \text{ or } t = \log(ax+b).$$

Then if $D = \frac{d}{dt}$, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{a}{ax+b}$ $\quad i.e.\ (ax+b)\frac{dy}{dx} = aDy.$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{(ax+b)}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

$i.e.\ (ax+b)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = a^2D(D-1)y.$ Similarly, $(ax+b)^3\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = a^3D(D-1)(D-2)y$ and so on.

After making these substitution in (4.12), that results a linear equation with constat coefficients.

**Example 4.15.** *Solve* $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x))$

**Sol.** This is a *Legendre's homogeneous linear equation.*
Put $1 + x = e^t$ *i.e.* $t = \log(1+x)$,
so that $(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} = Dy$ and $(1+x)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = D(D-1)y$, where $D = \frac{d}{dt}$.
Then given equation becomes

$$D(D-1)y + Dy + y = 2\sin(t). \quad \Rightarrow \quad (D^2+1)y = 2\sin(t) \tag{4.13}$$

which is linear equation with constant coefficients.

Its A.E. is $D^2 + 1 = 0$ whence $D = \pm i$. $\quad \therefore C.F. = c_1\cos(t) + c_2\sin(t)$.
And $P.I. = 2\frac{1}{D^2+1}\sin(t) = 2t\frac{1}{2D}\sin(t) = t\int\sin(t)dt = -t\cos(t)$.

[2] An French mathematician *Adrien Marie Legender* (1752-1833) who made important contribution to number theory, special functions and calculus of variation.

Hence the solution of (4.13) is $y = c_1\cos(t) + c_2\sin(t) - t\cos(t)$.
Put $t = \log(1+x)$, we get

$y = c_1\cos\big(\log(1+x)\big) + c_2\sin\big(\log(1+x)\big) - \log(1+x)\cos(\log(1+x))$ as the required solution of given equation.

**Exercise-IV**

Que :1 Solve the following differential equation.

1. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^3$ Ans . $c_1x + c_2x^2 + 0.5x^3$
2. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10(x + x^{-1})$ Ans . $y = c_1x^{-1} + x(c_2\cos(\log x) + c_3\sin(\log x)) + 5x + 2x^{-1}\log x$
3. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$ Ans . $y = c_1x^4 + c_2x^{-1} + (0.2)x^4\log x$
4. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = (1+x)^2$ Ans . $y = (c_1 + c_2\log x)x^2 + \frac{1}{4} + 2x + \frac{1}{2}x^2(\log x)^2$
5. $x\dfrac{d^2y}{dx^2} - 2x^{-1}y = x + x^{-2}$ Ans . $y = c_1x^2 + c_2x^{-1} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x$
6. $\dfrac{d^2y}{dx^2}x^{-1}\dfrac{dy}{dx} = 12x^{-2}\log x$ Ans . $y = c_1\log x + c_2 + 2(\log x)^3$
7. $(5+2x)^2\dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 2(2x+5)^2$ Ans . $y = (5+2x)^2\left[c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{\sqrt{2}}\right] - (5+2x)^2$
8. $(2x+3)^2\dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x$ Ans . $y = c_1(2x+3)^a + c_2(2x+3)^b - \frac{3}{14}(2x+3)$ where $a, b = \frac{3\pm\sqrt{57}}{4}$
9. $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{d^+y}{dx^+}4 = 4\cos(\log(1+x))$ Ans . $y = c_1\cos(t) + c_2\sin(t) + 2t\sin(t)$ where $t = \log(1+x)$
10. $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$ Ans . $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2)]$

**Definition 4.16.** *Polar Co-ordinates: Angle $\theta$ in polar co-ordinate system is directed angle, meaning angle can be positive or negative. Anticlockwise means positive, clockwise means negative.*
*In polar co-ordinate system, if $r$ is constant then a circle can be drawn and if $\theta$ is constant then a ray is obtained.*

$$\begin{aligned} P(r,\theta) &= P(-r,(2k+1)\pi\theta) \\ &= P(r,(2k\pi)\theta) \end{aligned}$$

*Advantage: Lesser things are required compared to cartesian co-ordinate system.*
*Disadvantage: In this system, same point has many co-ordinates.*

**Definition 4.17.** *Polar Co-ordinates in $R^2$*
*Let $O$ be a fixed point in the plane, let $\overrightarrow{OX}$ be a fixed ray in the plane. Then, for every point $P$ in the plane,*

*i one can find $r \geq 0$ such that $OP = r$ and*

*ii one can find $\theta \in [0, 2n]$ such that $m\angle POX = \theta$.*

*Here the ordered pair $(r, \theta)$ is called the polar co-ordinate of the point $P$. $O$ and $\overrightarrow{OX}$ are called the pole and the initial line respectively.*
*If $(r, \theta)$ is a polar co-ordinate of the point $P$, then $(r, 2k\pi + \theta)$, $(-r, \pi + \theta)$, $(-r, (2k+1)\pi\theta)$ are also polar co-ordinates of the same point $P$ for $\forall k \in Z$.*
*$r$ is called the radius vector and $\theta$ is called the angular co-ordinates of $P$.*

## 4.7 Relation between Cartesian and Polar Co-ordinates

Let $P(x, y)$ be a point in the cartesian co-ordinate plane. Take $O$ as the pole and $\overrightarrow{OX}$ as the initial line. Let $P(r, \theta)$ be the polar co-ordinate of $P$.

$$\begin{aligned}
&OP = |r| \\
&\implies OP^2 = r^2 \\
&\implies (x-O)^2 + (y-O)^2 = r^2 \\
&\implies x^2 + y^2 = r^2
\end{aligned} \tag{4.14}$$

Also, from the figure,

$$\begin{aligned}
&m\angle POM = \theta \\
&\implies \cos\theta = \tfrac{x}{r} \text{ and } \sin\theta = \tfrac{y}{r} \\
&\implies x = r\cos\theta \text{ and } y = r\sin\theta
\end{aligned} \tag{4.15}$$

**Example 4.18.** *Find the cartesian co-ordinates of the following polar points. Also plot the points*

*1* $(\sqrt{2}, \frac{\pi}{4})$

*2* $(2, \frac{\pi}{6})$

*3* $(2, \frac{-\pi}{3})$

*4* $(-2, \frac{-\pi}{4})$

**Sol.**

1 Here $A(\sqrt{2}, \frac{\pi}{4})$
$\therefore r = \sqrt{2}, \theta = \frac{\pi}{4}$
Now $x = r\cos\theta, y = r\sin\theta$
$\therefore x = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}(\frac{1}{\sqrt{2}})$ and $y = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}(\frac{1}{\sqrt{2}})$.
$\therefore (x, y) = (1, 1)$

2 Here $A(2, \frac{\pi}{6})$
$\therefore r = 2, \theta = \frac{\pi}{6}$
Now $x = r\cos\theta, y = r\sin\theta$
$\therefore x = 2\cos\frac{\pi}{6} = 2(\frac{\sqrt{3}}{2})$ and $y = 2\sin\frac{\pi}{6} = 2(\frac{1}{2})$.
$\therefore (x, y) = (\sqrt{3}, 1)$.

3 Here $A(2, \frac{-\pi}{3})$
$\therefore r = 2, \theta = \frac{-\pi}{3}$
Now $x = r\cos\theta, y = r\sin\theta$
$\therefore x = 2\cos\frac{-\pi}{3} = 2(\frac{1}{2})$ and $y = 2\sin\frac{-\pi}{3} = 2(-\frac{\sqrt{3}}{2})$.
$\therefore (x, y) = (1, -\sqrt{3})$.

4 Here $A(-2, \frac{-\pi}{4})$
$\therefore r = -2, \theta = \frac{-\pi}{4}$
Now $x = r\cos\theta, y = r\sin\theta$
$\therefore x = -2\cos\frac{-\pi}{4} = -2(\frac{1}{\sqrt{2}})$ and $y = -2\sin\frac{-\pi}{4} = -2(-\frac{1}{\sqrt{2}})$.
$\therefore (x, y) = (-\sqrt{2}, \sqrt{2})$.

**Example 4.19.** *Find polar co-ordinates of following cartesian points.*

*1* $(1, 1)$

*2* $(\sqrt{3}, 1)$

*3* $(-\sqrt{3}, -1)$

*4* $(-2, -2)$

***Sol.***

*1 Here* $(x, y) = (1, 1) \implies x = 1, y = 1$. *Now* $x^2 + y^2 = r^2 \implies r^2 = 1 + 1 \implies r = \sqrt{2}$.
*Now* $\cos\theta = \frac{x}{r}$ *and* $\sin\theta = \frac{y}{r}$.
$\therefore \cos\theta = \frac{1}{\sqrt{2}}$ *and* $\sin\theta = \frac{1}{\sqrt{2}}$. *Hence* $\theta = \frac{\pi}{4}$.
$\therefore (r, \theta) = (\sqrt{2}, \frac{\pi}{4})$.

*2 Here* $(x, y) = (\sqrt{3}, 1) \implies x = \sqrt{3}, y = 1$. *Now* $x^2 + y^2 = r^2 \implies r^2 = 3 + 1 \implies r = 2$.
*Now* $\cos\theta = \frac{x}{r}$ *and* $\sin\theta = \frac{y}{r}$.
$\therefore \cos\theta = \frac{\sqrt{3}}{2}$ *and* $\sin\theta = \frac{1}{2}$. *Hence* $\theta = \frac{\pi}{6}$.
$\therefore (r, \theta) = (2, \frac{\pi}{6})$.

3 *Here* $(x,y)=(-\sqrt{3},-1) \implies x=-\sqrt{3}, y=-1$. *Now* $x^2+y^2=r^2 \implies r^2=3+1 \implies r=2$.
*Now* $\cos\theta=\frac{x}{r}$ *and* $\sin\theta=\frac{y}{r}$.
$\therefore \cos\theta=-\frac{\sqrt{3}}{2}$ *and* $\sin\theta=-\frac{1}{2}$. *Hence* $\theta=\frac{7\pi}{6}$.
$\therefore (r,\theta)=(2,\frac{7\pi}{6})$.

4 *Here* $(x,y)=(-2,-2) \implies x=-2, y=-2$. *Now* $x^2+y^2=r^2 \implies r^2=4+4 \implies r=2\sqrt{2}$.
*Now* $\cos\theta=\frac{x}{r}$ *and* $\sin\theta=\frac{y}{r}$.
$\therefore \cos\theta=-\frac{2}{2\sqrt{2}}$ *and* $\sin\theta=-\frac{2}{2\sqrt{2}}$. $\therefore \cos\theta=-\frac{1}{\sqrt{2}}$ *and* $\therefore \sin\theta=-\frac{1}{\sqrt{2}}$. *Hence* $\theta=\frac{5\pi}{4}$.
$\therefore (r,\theta)=(2\sqrt{2},\frac{5\pi}{4})$.

**Theorem 4.20.** *Find distance formula in polar co-ordinate system in* $R^2$.

*Proof.* Let $A(r_1,\theta_1)$ and $(r_2,\theta_2)$ are two points in polar co-ordinate systems.
The cartesian co-ordinates of $A$ and $B$ are $A(r_1\cos\theta_1, r_1\sin\theta_1)$, $B(r_2\cos\theta_2, r_2\sin\theta_2)$. Now

$$\begin{aligned}
AB &= \sqrt{(r_1\cos\theta_1-r_2\cos\theta_2)^2+(r_1\sin\theta_1-r_2\sin\theta_2)^2}\\
&= \sqrt{r_1^2\cos^2\theta_1-2r_1r_2\cos\theta_1\cos\theta_2+r_2^2\cos^2\theta_2+r_1^2\sin^2\theta_1+r_2^2\sin^2\theta_2-2r_1r_2\sin\theta_1\sin\theta_2}\\
&= \sqrt{r_1^2+r_2^2-2r_1r_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}\\
AB &= \sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}
\end{aligned}$$

□

**Theorem 4.21.** *Obtain the formula for the area of* $\triangle ABC$ *in polar co-ordinate system.*

*Proof.* Let $A(r_1,\theta_1), B(r_2,\theta_2)$ and $C(r_3,\theta_3)$ be the vertices of the $\triangle ABC$. Hence the cartesian co-ordinate $A,B$ and $C$ are $A(r_1\cos\theta_1, r_1\sin\theta_1)$, $B(r_2\cos\theta_2, r_2\sin\theta_2)$ and $C(r_3\cos\theta_3, r_3\sin\theta_3)$.

$$\triangle ABC=\frac{1}{2}\begin{vmatrix} r_1\cos\theta_1 & r_1\sin\theta_1 & 1\\ r_2\cos\theta_2 & r_2\sin\theta_2 & 1\\ r_3\cos\theta_3 & r_3\sin\theta_3 & 1\end{vmatrix}$$

□

**Theorem 4.22.** *Obtain the equation of line passing through* $A(r/-1,\theta_1)$ *and* $B(r_2,\theta_2)$.

*Proof.* The cartesian co-ordinates of $A$ and $B$ are $A(r_1\cos\theta_1, r_1\sin\theta_1)$, $B(r_2\cos\theta_2, r_2\sin\theta_2)$. The cartesian equation of $\overleftrightarrow{AB}$ is

$$\begin{vmatrix} x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}=0$$

In polar co-ordinates, the equation of $\overleftrightarrow{AB}$ is,

$$\begin{vmatrix} r\cos\theta & r\sin\theta & 1 \\ r_1\cos\theta_1 & r_1\sin\theta_1 & 1 \\ r_2\cos\theta_2 & r_2\sin\theta_2 & 1 \end{vmatrix} = 0$$

$$\implies (r_1\cos\theta_1 r_2\sin\theta_2 - r_1 r_2\cos\theta_2\sin\theta_1) - (r\cos\theta r_2\sin\theta_2 - r r_2\cos\theta_2\sin\theta) + (r r_1\cos\theta\sin\theta_1 - r r_1\cos\theta_1\sin\theta) = 0$$

$$\implies r_1 r_2\sin(\theta_2-\theta_1) - r r_2\sin(\theta_2-\theta) + r r_1\sin(\theta_1-\theta) = 0$$

$$\implies \frac{\sin(\theta_2-\theta_1)}{r} = \frac{\sin(\theta_2-\theta)}{r_1} - \frac{\sin(\theta_1-\theta)}{r_2}$$

$$\implies \frac{\sin(\theta_1-\theta_2)}{r} = \frac{\sin(\theta-\theta_2)}{r_1} - \frac{\sin(\theta-\theta_1)}{r_2}$$

is the polar equation of a line passing through $A(r_1,\theta_1)$ and $B(r_2,\theta_2)$. □

**Theorem 4.23.** *Obtain the polar equation of a line in $p-\alpha$ form.*

*Proof.* Let $p$ be the perpendicular distance from the pole to a line $L$ in the polar plane. Draw $\overline{OM} \perp L, M \in L$. Let $m\angle MOX = \alpha$. The polar co-ordinates of $M$ is $M(p,\alpha)$.
Let $P(r_1\theta)$ be a point on the line $L$ other than $M$.
$\therefore OP$ distance is $r$ and $m\angle POX = \theta$.
$\therefore m\angle POM = \theta-\alpha$ or $\alpha-\theta = \pm\theta-\alpha = |\theta-\alpha|$. From the right-angled $\triangle POM$,

$$\cos(\angle POM) = \frac{OM}{OP}$$
$$\implies OM = OP\cos(\pm\theta-\alpha)$$
$$\implies p = r\cos(\theta-\alpha) \quad (\because \cos\theta = \cos(-\theta))$$

which is the required equation. □

## 4.8 Deductions:

1 If $O \in L$, then $P = O$. Hence $r\cos(\theta-\alpha) = 0$. That is if pole is on line $L$, then $r\cos(\theta-\alpha) = 0$ is the equation of line passing through pole.

2 If $L \perp \overleftrightarrow{OX}$, then $\alpha = 0$. Hence $p = r\cos(\theta-0)$.
$\therefore p = r\cos\theta$.

3 If $L \parallel \overleftrightarrow{OX}$, then $\alpha = \frac{\pi}{2}$. Hence $p = r\cos(\theta-\frac{\pi}{2}) = r\sin\theta$. Equation of line will be $p = r\sin\theta$.

4 If $L = \overleftrightarrow{OX}$, then $p = 0$ and $\alpha = \frac{\pi}{2}$. Hence the equation of line will be $r\sin\theta = 0$.

**Example 4.24.** *Prove that the points* $(6,0), (3,\frac{pi}{2})$ *and* $(-3,\frac{7\pi}{3})$ *are non-collinear.*

**Sol.** The polar equation of a line passing through $(6,0), (3, \frac{\pi}{3})$ and $(-3, \frac{7\pi}{3})$ is,

$$\begin{vmatrix} r\cos\theta & r\sin\theta & 1 \\ r_1\cos\theta_1 & r_1\sin\theta_1 & 1 \\ r_2\cos\theta_2 & r_2\sin\theta_2 & 1 \end{vmatrix} = \begin{vmatrix} 6(1) & 6(0) & 1 \\ 3(\frac{1}{2}) & 3(\frac{\sqrt{3}}{2}) & 1 \\ -3(\frac{1}{2}) & -3(\frac{\sqrt{3}}{2}) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 0 & 1 \\ \frac{3}{2} & 3\frac{\sqrt{3}}{2} & 1 \\ -\frac{3}{2} & -3\frac{\sqrt{3}}{2} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 0 & 1 \\ \frac{3}{2} & 3\frac{\sqrt{3}}{2} & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2(18\frac{\sqrt{3}}{2})$$

$$= 18\sqrt{3}$$

$$\neq 0$$

$\therefore$ the given points are non-collinear

**Example 4.25.** *Obtain the polar co-ordinates of the foot of*